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Waves in liquids with vapour bubbles

By R. I. NIGMATULIN,
Institute for North Development Problems, Siberian Branch of the USSR Academy of Sciences, 625003, Tyumen, box. 2774, USSR

N. S. KHABEEV
Institute of Mechanics, M.V. Lomonosov Moscow University, Moscow V-234, USSR

AND ZUONG NGOK HAI
Institute of Mechanics, The National Centre for Scientific Research of the Socialist Republic of Vietnam, Hanoi

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An investigation of wave processes in liquids with vapour bubbles with interphase heat and mass transfer is presented. A single-velocity two-pressure model is used which takes into account both the liquid radial inertia due to medium volume changes, and the temperature distribution around the bubbles. An analysis of the microscopic fields of physical parameters is aimed at closing the system of equations for averaged characteristics. The original system of differential equations of the model is modified to a form suitable for numerical integration. An elliptic equation is obtained to determine the field of the mixture average pressure at an arbitrary time through the known fields of the remaining quantities. The existence of the steady structure of shock waves, either monotonic or oscillatory, is proved. The effect of the initial conditions, shock strength, volume fraction, and dispersity of the vapour phase and of the thermophysical properties of the phases on shock-wave structure and relaxation time is studied. The influence of nonlinear, dispersion and dissipative effects on the wave evolution is also investigated. The shock adiabat for reflected waves is analysed. The results obtained have proved that the interphase heat and mass transfer determined by the thermal diffusivity of the liquid greatly influences the wave structure. The possible enhancement of disturbances in the region of their initiation is shown. The model has been tested for suitability and the results of calculations have been compared with experimental data.

1. Introduction

Vapour-liquid media are interesting from the viewpoint that owing to nonlinear, dispersion and dissipative effects the wave patterns in them may be diverse and easily altered by varying the hydrodynamic conditions, structure and thermophysical properties of the mixture, as well as the interphase interaction (mass, momentum and energy). The distinctive features exhibited by a bubbly liquid in dynamic processes are connected with local deformation inertia, when the volume of the medium is changed owing to a change in the bubble volume, as well as with the elasticity of compression of the gas inside the bubbles. The above inertia, which may be viewed as the inertia of the added liquid mass to the bubble in its radial motion described by the Rayleigh–Plesset equation, and elasticity of compression lead to non-holonomicity of the equation of state containing the medium pressure p which depends not only on the medium density ρ but also on its first and second derivatives, $\dot{\rho}$ and $\ddot{\rho}$:

$$p = p(\rho, \dot{\rho}, \ddot{\rho}, p_g),$$

where p_g is the gas pressure in the bubbles. Kogarko (1961) was the first to report the dependence of p on $\ddot{\rho}$.

Liquid deformation inertia and gas elasticity of compression lead to bubble pulsations and, eventually, to oscillatory waves (Batchelor 1969; Wijngaarden 1970). The propagation of pressure waves in bubbly media may cause vapour-phase condensation (condensation waves), thereby radically altering the physical structure of the medium.

Shock waves in liquids with bubbles of an insoluble and non-condensable (inert) gas have been investigated theoretically and experimentally (Batchelor 1969; Wijngaarden 1970, 1972; Kutateladze *et al.* 1972; Gel'fand *et al.* 1973; Noordzij 1973; Noordzij & Wijngaarden 1974; Nigmatulin, Khabeev & Shagapov 1974; Nigmatulin & Shagapov 1974; Gubaidulin, Ivandaev & Nigmatulin 1976; Aidagulov, Khabeev & Shagapov 1977; Kuznetsov *et al.* 1978; Drumheller, Kipp & Bedford 1982; Tan & Bankoff 1984).

An analysis has been undertaken (Nigmatulin *et al.* 1974; Nigmatulin & Shagapov 1974) that allows for non-equilibrium effects caused by the inertia and viscosity of the liquid in its radial motion around the bubbles and by the translational motion of the bubbles relative to the carrier liquid, as well as by a finite thermal conductivity of the gas (thermal non-equilibrium). It is shown that the structure of the steady wave in a gas–liquid mixture with bubbles of constant mass has an appreciable thickness (for bubbles with a radius of $\sim$ 1 mm this thickness is of the order of 1 m), and that for not too viscous liquids the two-velocity inertial and viscous effects due to bubble longitudinal motion relative to the liquid are hardly noticeable against the background of the interphase heat-transfer effects determined by the gas thermal diffusivity.

An investigation of transient waves has revealed the effect of shock-wave enhancement, when a shock pulse is divided into solitary waves (solitons) the amplitudes of which may greatly exceed the initial pulse amplitude (Nigmatulin 1982). The effect results from the previously mentioned local deformation inertia of a bubbly liquid which can compress even though the loading has already been removed. This effect depends strongly on the signal duration, volume concentration of the bubbles and, strange as it may seem, on the properties of the gas.

The propagation of waves in a liquid with vapour bubbles has been investigated by Trammell (1962), Nakoryakov & Shreiber (1979), Pokusaev (1979), Azamatov & Shagapov (1981), Nakoryakov *et al.* (1984), Nigmatulin *et al.* (1982) and Zuong Ngok Hai, Nigmatulin & Khabeev (1982, 1984). In work by Borisov *et al.* (1977), Deksnis (1978), Gel'fand *et al.* (1978) and Borisov *et al.* (1982) the effect of a strong local increase in the intensity of the waves propagating through the mixture was experimentally observed and it was discovered that waves in these media may have a peaky structure. Borisov *et al.* (1977, 1982) showed that the anomalous growth of pressure in a bubbly vapour–liquid medium during wave propagation is connected with fragmentation of large bubbles into numerous small ones and a rapid condensational collapse of the latter. As a result, the kinetic energy of the liquid radial motion around the bubbles is transformed into the energy of compression of the liquid.

Nigmatulin, Khabeev & Nagiev (1979), Zuong Ngok Hai *et al.* (1982, 1984), Nigmatulin *et al.* (1982) proved theoretically that in these media, owing to radial motion around bubbles, the waves may have a peaky structure; and the effect of a pressure increase in the wave was estimated in terms of the cell model. A review of modern problems connected with shock-wave propagation in bubbly vapour–liquid media is given elsewhere (Nigmatulin 1982; Gubaidullin *et al.* 1982).

In the present work the evolution of non-stationary shock waves of moderate intensities and pulse disturbances of finite lengths in a liquid with vapour bubbles has been studied on the basis of the complete system of differential equations. The structure of a steady or limiting shock wave, formed as a result of a steady or sufficiently prolonged action on the mixture, has been investigated. The reflection of shock waves from a rigid wall has been considered on the basis of the shock-wave discontinuity equations.

2. Basic equations

Wave processes in a bubbly liquid are considered here using continuum-mechanics methods under the following basic assumptions:

(i) the distances over which the flow parameters (for example, oscillatory wavelengths) vary significantly are much larger than the distances between the bubbles, which are themselves much larger than the bubble diameters (i.e. the volume fraction of the vapour phase is small enough, $\alpha_v \lesssim 0.1$);

(ii) the mixture is locally monodispersed, i.e. in each material volume all the bubbles are spherical and of the same radius;

(iii) viscosity and thermal conduction are important only in the processes of interphase interaction and, in particular, in bubble pulsations;

(iv) nucleation, fragmentation, interaction and coagulation of the bubbles are absent;

(v) the velocities of the macroscopic motion of the phases coincide.

The last assumption allows us to describe bubble volume changes, temperature distributions around the bubbles, condensation and evaporation in terms of the spherically symmetrical model using the equations for bubble radial pulsations and radial thermal conduction of the liquid. This assumption originates from the fact that for vapour bubbles the role of the interphase heat and mass transfer becomes greater than for gas bubbles, and the two-velocity effects are therefore less significant on the background of thermal dissipation (Nigmatulin *et al.* 1974; Zuong Ngok Hai *et al.* 1982). As far as suspensions in gases are concerned, two-velocity effects play a decisive role in wave dynamics, whereas in bubbly liquids these effects are insignificant because the relative velocity of the bubbles is small if compared with the velocities of the phases (Nigmatulin & Shagapov 1974; Nigmatulin 1982; see also experiments by Kalra & Zvirin 1981). Nevertheless, it should be borne in mind that the interphase heat and mass transfer in strong shock waves may be more intense than predicted by the spherically symmetrical model owing to an increase in the interfacial surface caused by deformation and fragmentation of the bubbles.

Under the assumptions listed above the vapour–liquid medium can be considered within the framework of a model of two interacting and interpenetrating continuous media, viz. the carrier liquid and the vapour phase (Nigmatulin 1978). In the Lagrangian system of coordinates (ξ, t) the equations of conservation of mass, bubble number density and of momentum of the mixture for one-dimensional motion are as follows:

$$\left.\begin{aligned}&\frac{\partial \rho_\ell}{\partial t}+\frac{\rho\rho_\ell}{\rho_0}\frac{\partial v}{\partial \xi}=-4\pi R^2 nj; \quad \frac{\partial \rho_{\mathrm{v}}}{\partial t}+\frac{\rho\rho_{\mathrm{v}}}{\rho_0}\frac{\partial v}{\partial \xi}=4\pi R^2 nj;\\ &\frac{\partial n}{\partial t}+\frac{\rho n}{\rho_0}\frac{\partial v}{\partial \xi}=0; \quad \frac{\partial v}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial \xi}=g;\end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned}&\rho_i=\alpha_i\rho_i^0; \quad \alpha_\ell+\alpha_{\mathrm{v}}=1; \quad \rho=\rho_\ell+\rho_{\mathrm{v}}; \quad \alpha_{\mathrm{v}}=\tfrac{4}{3}\pi R^3 n;\\ &p=\alpha_\ell p_\ell+\alpha_{\mathrm{v}}\left(p_{\mathrm{v}}-\frac{2\sigma}{R}\right); \quad (i=\ell,\mathrm{v}),\end{aligned}\right\} \tag{2}$$

where the subscripts $i=\ell$, v refer to the parameters of the liquid and vapour, respectively, the subscript 0 referring to the parameters of the initial equilibrium state; $\alpha_i, p_i, \rho_i, \rho_i^0$ are the volume fraction, pressure, mean and true densities of the ith phase, respectively, v is the longitudinal velocity, n is the number of bubbles per unit volume, R is the bubble radius, j is the rate of phase transition per unit interfacial surface ($j>0$ for evaporation and $j<0$ for condensation), g is the intensity of the external mass forces, σ is the coefficient of surface tension, ξ is the Lagrangian longitudinal coordinate and t is the time. In this case the relationship between the Eulerian and Lagrangian coordinates is expressed by the following relationship (Sedov 1984):

$$y(\xi,t)=\int_0^t v(\xi,\tau)\,\mathrm{d}\tau.$$

Note that, by virtue of assumption (i) ($\alpha_{\mathrm{v}} \ll 1$), from (2) it follows that the average pressure p in the mixture practically coincides with the pressure in the liquid phase ($p \approx p_\ell$).

An equation for a change in the mass of an individual bubble can be obtained from the equations of conservation of the mass of the vapour phase and the bubble number density (1):

$$\frac{\partial}{\partial t}\left(\tfrac{4}{3}\pi R^3\rho_{\mathrm{v}}^0\right)=4\pi R^2 j.$$

The system of hydrodynamic equations (1) will be closed if the equation of state, the condition of the simultaneous deformation of the phases and the equation for determining the phase transition rate j are assigned. The evolution of pressure waves of moderate intensities can be considered under the following additional assumptions:

(*a*) the carrier liquid phase is incompressible:

$$\rho_\ell^0=\mathrm{const.} \tag{3}$$

(*b*) the vapour obeys the equation of state of a perfect gas, and being in the saturated state at the interface it obeys the Clapeyron–Clausius equation

$$p_{\mathrm{v}}=B\rho_{\mathrm{v}}^0 T_{\mathrm{v}}; \quad \frac{\mathrm{d}T_{\mathrm{v}\sigma}}{\mathrm{d}p_{\mathrm{v}}}=\frac{T_{\mathrm{v}\sigma}}{l\rho_{\mathrm{v}\sigma}^0}\left(1-\frac{\rho_{\mathrm{v}\sigma}^0}{\rho_\ell^0}\right). \tag{4}$$

Here T is the absolute temperature, B is the gas constant, l is the specific heat of vaporization, the symbol σ referring to the parameters at the interface.

The assumption that the carrier liquid is incompressible is valid when the wave velocity U, relative to the medium before the front, and the volume fraction of the vapour phase satisfy the conditions $(U/a_\ell)^2 \ll 1$ and $\alpha_{\mathrm{v}} \gg \alpha_{\mathrm{c}}=p_0/\rho_\ell^0 a_\ell^2$, respectively (Nigmatulin & Shagapov 1974). In this case the mixture is compressed at the expense of the compression of the vapour in the bubbles. Under normal conditions, $p \sim 0.1$

MPa, the above conditions hold for most liquids when the volume fraction of the vapour phase $\alpha_v \gtrsim 10^{-2}$.

Transfer processes in a two-phase mixture are determined by the distributions of microparameters near inhomogeneities. For an analysis not to be too complicated, one has to make use of models that could considerably simplify the microprocess equations. A possible model is one that employs the concept of a cell with a test bubble in it at any point specified by a vector $\boldsymbol{y}$. The cell dimensions are determined by the volume fractions of the phases and equal to $R\alpha_v^{-\frac{1}{3}}$, the cell centre coinciding with the centre of the test bubble. This cell moves with the macroscopic velocity $\boldsymbol{v}(t,\boldsymbol{y})$ of the vapour phase at the point considered. The distributions of microparameters inside a cell are described by the equations for the corresponding microprocesses with the boundary conditions on the test bubble surface (which determine the interphase interaction) and on the external boundary of the cell (which determine the action of the external, relative to the cell, carrying phase).

Consider a spherically symmetric test bubble with its centre at a point y, the microparameters (marked by primes) inside and around the bubble being dependent on time t, the position of the bubble centre y, and the distance r of a microparticle from the centre (or only on t and r in the Lagrangian system of coordinates moving together with the bubble with velocity v):

$$T'_v = T'_v(t,y,r); \quad \rho_v^{0\prime} = \rho_v^{0\prime}(t,y,r); \quad T'_\ell = T'_\ell(t,y,r).$$

In spite of the non-uniformity of the vapour temperature and density fields inside the bubbles we consider the vapour pressure field to be uniform (homobaric conditions), which is valid provided that $(w_{v\sigma}/a_v)^2 \ll 1$ (Nigmatulin 1978). Here $w_{v\sigma}$ is the mass velocity of the vapour on the bubble surface, a_v is the velocity of sound in the vapour.

To determine the temperature, density and heat-flux distributions, we use the equations of discontinuity, heat conduction and the equation of state. The vapour and the liquid at the interface are assumed to be in thermodynamic equilibrium. The phase transition rate j may be found from the boundary conditions on the bubble surface. The boundary conditions at the bubble centre may be determined from the condition of finite heat flux, temperature and density. In the absence of a macroscopic heat flux in the carrying phase the condition on the cell boundary should reflect the cell adiabaticity. The system of equations describing the distributions of the microparameters inside and around the test bubble and the boundary conditions in the system of coördinates (t,ξ,r) are as follows:

$$\left.\begin{array}{ll}
\rho_\ell^0 c_\ell\left(\dfrac{\partial T'_\ell}{\partial t}+w_{\ell\sigma}\dfrac{R^2}{r^2}\dfrac{\partial T'_\ell}{\partial r}\right)=\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(\lambda_\ell r^2\dfrac{\partial T'_\ell}{\partial r}\right) & (r>R); \\[2ex]
\rho_v^{0\prime} c_{p_v}\left(\dfrac{\partial T'_v}{\partial t}+w'_v\dfrac{\partial T'_v}{\partial r}\right)=\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(\lambda_v r^2\dfrac{\partial T'_v}{\partial r}\right)+\dfrac{\partial p_v}{\partial t} & (r<R); \\[2ex]
\dfrac{\partial \rho_v^{0\prime}}{\partial t}+\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\rho_v^{0\prime} w'_v r^2)=0; & \\[2ex]
\left.\begin{array}{l}
T'_\ell = T'_v = T_s(p_v); \quad jl = -(q_{\ell\sigma}+q_{v\sigma}) \\
q_{\ell\sigma} = -\lambda_\ell \left.\dfrac{\partial T'_\ell}{\partial r}\right|_R, \quad j=\rho_\ell^0\left(\dfrac{\partial R}{\partial t}-w_{\ell\sigma}\right)=\rho_v^0\sigma\left(\dfrac{\partial R}{\partial t}-w_{v\sigma}\right)
\end{array}\right\} & (r=R(t)) \\[2ex]
w'_v=0, \quad \dfrac{\partial T'_v}{\partial r}=0 \quad (r=0); \quad \dfrac{\partial T'_\ell}{\partial r}=0 \quad (r=\alpha_v^{-\frac{1}{3}}R(t)), &
\end{array}\right\} \quad (5)$$

where w' is the velocity of radial motion, c_{pv} is the specific heat at constant pressure of the vapour, λ is the thermal conductivity, $q_{\ell\sigma}$ and $q_{v\sigma}$ are the heat fluxes to the liquid and vapour, respectively, from the interface. Subscript s refers to saturation.

It should be noted that an allowance for the microparameter distributions makes the solution to the problems of wave dynamics in bubbly liquids much more difficult. Even the simplified account of the temperature microdistribution inside and around the bubbles for a one-dimensional unsteady flow of bubbly mixtures (under the assumption of a spherical symmetry of the microprocess, and with no allowance for the influence of the relative motion of the bubbles) considered here leads to the necessity of solving a system of partial differential equations in three independent variables.

We note a principal difference in the description of heat exchange between a 'cold' liquid and gaseous bubbles without phase transitions on the one hand, and between a 'hot' liquid and vapour bubbles with condensation and evaporation on the other hand. In the case of gaseous bubbles in a 'cold' liquid the change of pressure $p_v(t)$ in the bubble depends on interphase heat exchange. In this case $|q_{v\sigma}| = |q_{\ell\sigma}|$ and heat exchange is determined by the heat resistance of the gas but not of the liquid, i.e. by the internal heat problem (Chapman & Plesset 1971; Nigmatulin & Khabeev 1974; Plesset & Prosperetti 1977; Nigmatulin, Khabeev & Nagiev 1981). In the case of vapour bubbles in a 'hot' liquid the change in $p_v(t)$ depends primarily on the intensity of evaporation and condensation $j = -(q_{\ell\sigma}+q_{v\sigma})/l$. In this case $|q_{v\sigma}| \ll |q_{\ell\sigma}|$ because under conditions which are far from critical the following estimates are usually valid:

$$\rho_\ell^0 \gg \rho_v^0, \quad \lambda_\ell \gg \lambda_v, \quad \mathscr{D}_v \gg \mathscr{D}_\ell,$$

where D is thermal diffusivity

$$\mathscr{D}_v = \frac{\lambda_v}{\rho_v^0 c_{pv}}, \quad \mathscr{D}_\ell = \frac{\lambda_\ell}{\rho_\ell^0 c_\ell}.$$

Therefore there is no need for the exact calculation of $q_{v\sigma}$ because the process is mainly determined by $q_{\ell\sigma}$, the calculation of which requires that nonlinear equation (5) of non-stationary heat conduction in liquid be solved. It thus follows that the evolution of pressure waves in a 'hot' liquid with vapour bubbles is determined by the thermal diffusivity of the liquid rather than of the gas.

For this reason the uniform-bubble model is widely used in investigations on the dynamics of vapour bubbles in a 'hot' liquid, which corresponds to the asymptotic case of the equation of heat conduction in the vapour (second equation in (5)) at $D_v \to \infty$ or, more precisely, when the wavelength of the 'thermo-diffusion wave' in the vapour D_v/w_v is far larger than the bubble radius R:

$$\frac{\mathscr{D}_v}{w_v R} \gg 1.$$

Even in cases when the above condition does not hold true ($D_v/w_v R \gtrsim 1$) the uniform-bubble model, which leads to significant errors in calculating $q_{v\sigma}$, does not give appreciable errors in calculating j and $p_v(t)$ because $|q_{v\sigma}| \ll |q_{\ell\sigma}|$.

The applicability of the model of a uniform vapour bubble has been investigated in detail (Nigmatulin & Khabeev 1975; Nagiev & Khabeev 1981). A comparison of the calculations made with allowance for the bubble-temperature non-uniformity

($D_v/w_v R \sim 1$) with the calculations in terms of the uniform-bubble model has shown that taking account of temperature non-uniformity and the refinement of $q_{v\sigma}$ have an insignificant effect on the bubble dynamics, provided the conditions are far from the critical ones.

At high parameters values when the thermophysical properties of the gas and vapour become close, and for regimes when $D_v/w_v R \ll 1$ the assumption about the bubble uniformity is unjustified and may lead to significant errors. This assumption is not important when the problem is solved numerically and is given here as a possible way of simplifying the calculations when the exact calculation of $q_{v\sigma}$ is not necessary.

The variants presented in the paper were also calculated without using this simplification.

Within the framework of a uniform bubble containing a saturated vapour $T_v = T_s(p_v(t))$, the heat flux $q_{v\sigma} = \lambda_v \partial T'_v/\partial r|_R$ spent on a change in vapour saturation temperature caused by a pressure changes is non-zero because the uniform-bubble model corresponds to the asymptotic condition $D_v \to \infty$, $\partial T_v/\partial r \to 0$. In this case $D_v \partial T_v/\partial r \neq 0$ because we have an indeterminacy of the type $\infty \times 0$.

For $q_{v\sigma}$ to be calculated in terms of the model of a uniform bubble filled with saturated vapour, we use the equation of the heat flowing to the vapour phase (the second equation in system (5)). Substituting the total derivative of the saturated-vapour temperature by the derivative of the pressure according to (4), and integrating this equation with respect to r within the limits from 0 to R we arrive at

$$q_{v\sigma} = \tfrac{1}{3}R\left[\frac{c_{pv}T_v}{l}\left(1-\frac{\rho_v^0}{\rho_\ell^0}\right)-1\right]\frac{\partial p_v}{\partial t} = \frac{c_S T_v}{l}\tfrac{1}{3}R\frac{\partial p_v}{\partial t}, \tag{6}$$

where c_S is the vapour specific heat along the phase equilibrium curve (Landau & Lifshitz 1976):

$$c_S = c_p - T\left(\frac{\mathrm{d}p}{\mathrm{d}T}\right)_S \frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)_p.$$

For most liquids, particularly for water, under normal conditions ($p \sim 0.1$ MPa) $c_S < 0$. This means that for vapour to remain in a saturated state when it is compressed, heat should be abstracted from it. For water $c_S = 0$ at $p \sim 3$ MPa.

The pressures of the phases and the bubble radius are related by the condition of simultaneous deformation, as described by the Rayleigh–Plesset equation

$$\left.\begin{aligned} R\frac{\mathrm{d}w_{\ell\sigma}}{\mathrm{d}t}+\tfrac{3}{2}w_{\ell\sigma}^2+\frac{4\nu_\ell}{R}w_{\ell\sigma} &= \frac{p_v-p_\ell-2\sigma/R}{\rho_\ell^0}, \\ \frac{\mathrm{d}R}{\mathrm{d}t} = w_{\ell\sigma}+\frac{j}{\rho_\ell^0} &= w_{v\sigma}+\frac{j}{\rho_{v\sigma}^0}, \end{aligned}\right\} \tag{7}$$

where ν_ℓ is the kinematic viscosity.

In terms of the cell model (Nigmatulin 1978, 1979) corrections for the 'non-singleness' of the bubble may be introduced into (7). These corrections characterize the difference of the fictitious pressure p_∞ at infinity from the average pressure p_ℓ in the liquid. Numerical calculations have shown that the introduction of these corrections has a slight effect on the structure of the shock wave in a bubbly medium (Zuong Ngok Hai *et al.* 1982).

Using (4)–(7), the equation for a change in the mass of an individual bubble can be written in the following form:

$$\left.\begin{aligned} \frac{\partial p_{\mathrm{v}}}{\partial t} &= -\frac{3\gamma p_{\mathrm{v}}}{R\gamma_*}\left(\frac{q_{\ell\sigma}}{l\rho_{\mathrm{v}}^0}+\frac{\partial R}{\partial t}\right); \\ \gamma_* &= 1+(\gamma-1)\left(1-\frac{c_{p\mathrm{v}}T_{\mathrm{v}}}{l}\right)\left[1-\frac{c_{p\mathrm{v}}T_{\mathrm{v}}}{l}\left(1-\frac{\rho_{\mathrm{v}}^0}{\rho_\ell^0}\right)\right]; \end{aligned}\right\} \tag{8}$$

where γ is the specific heat ratio.

Let us transform (1) to a form suitable for numerical integration. From the equations of conservation of the masses of the phases and bubble number density one can obtain, subject to assumption (iii), differential relationships for determining the velocity and density of the mixture

$$\frac{\partial v}{\partial \xi} = 3\,\frac{\alpha_{\mathrm{v}}\rho_0 w_{\ell\sigma}}{\rho R}; \quad \frac{\partial \rho}{\partial t} = -3\,\frac{\alpha_{\mathrm{v}}\rho w_{\ell\sigma}}{R}. \tag{9}$$

By differentiating the equation of conservation of momentum with respect to ξ, the first equation in (9) with respect to t and equating the mixed derivatives, $\partial^2 v/\partial t\,\partial\xi$ and $\partial^2 v/\partial\xi\,\partial t$, we obtain, if (7) is taken into account, the following differential equation for the average pressure:

$$\frac{\partial^2 p}{\partial \xi^2}+M(\xi)\,\frac{\partial p}{\partial \xi} = K; \tag{10}$$

where

$$M(\xi) = -\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial \xi};$$

$$K = -\frac{3\alpha_{\mathrm{v}}\rho_0^2}{\rho R^2}\left[\tfrac{1}{2}w_{\ell\sigma}^2-\frac{4\nu_\ell}{R}\,w_{\ell\sigma}+\frac{p_{\mathrm{v}}-p_\ell-2\sigma/R}{\rho_\ell^0}\right].$$

It represents an elliptic equation. Note that, according to this equation, pressure disturbances propagate with an infinite velocity. This is the consequence of the incompressibility of the carrier liquid which transmits pressure disturbances. The influence of the bubbles and vapour properties is exhibited through the function $K = K(R_0, w_{\ell\sigma}, p_{\mathrm{v}}, p)$ in which R_0, $w_{\ell\sigma}$ and p_{v} can be determined from (7) and (8). The infinite velocity of disturbance propagation in the carrier liquid is the frozen sound speed in a given two-phase dispersion medium (the phase velocity of sound depends on the frequency of disturbances).

Equations (9) and (10) allow us to determine the velocity and pressure fields of the mixture at fixed instants through the known fields of the remaining parameters.

3. Results and discussion

To investigate the main regularities of the propagation of plane non-stationary shock waves and the evolution of pulse disturbances in vapour–liquid bubbly media, we used the closed system of equations (2)–(10). The corresponding mathematical problems consisted in finding solutions of the system (2)–(10) (by numerical integration), subject to the following initial and boundary conditions at specified cross-sections for the volume of mixture chosen ($\xi = 0$, $\xi = L$):

$$\left.\begin{array}{ll}\left.\begin{array}{l} p_\ell = p_0; \quad p_v = p_0 + \dfrac{2\sigma}{R_0}; \quad R = R_0; \\ v = w_{\ell\sigma} = w_{v\sigma} = 0; \quad T_\ell = T_v = T_0; \end{array}\right\} & (t = 0); \\ p = f_b(t) \quad \text{or} \left.\dfrac{\partial p}{\partial \xi}\right|_{\xi=0} = \phi_b(t) & (\xi = 0), \\ p = f_L(t) \quad \text{or} \left.\dfrac{\partial p}{\partial \xi}\right|_{\xi=L} = \phi_L(t) & (\xi = L). \end{array}\right\} \tag{11}$$

The propagation of short pressure delta-pulses (similar to those formed during explosions of microcharges) was studied. The pulses were modelled by assigning different laws of a rapid pressure change which correspond to a linear rise and a linear drop of the pressure in the zone of decreasing pressure. For a detla-pulse the function $f(t)$ is of the form

$$f_b(t) = \begin{cases} p_0(1 + b_1 t); & t < t_1, \\ p_0[(1 + b_1 t_1) - b_2(t - t_1)]; & t_1 \leqslant t \leqslant t_2, \\ p_0; & t_2 < t, \end{cases}$$

where the constants t_1, t_2 are determined by the duration of the initial pulse, and the non-negative coefficients b_1, b_2 by its intensity. The case when external mass forces are absent, i.e. $g = 0$, is considered. However, an allowance for these forces, when numerically solving the system of (2)–(10), offers no difficulties.

The system of equations (2)–(10) was solved by combining the modified Euler method with the method of factorization. The solution of this multi-parameter problem is determined by the following dimensionless numbers and combinations:

$$\left.\begin{array}{l} G = \dfrac{gR_0}{a_*^2(\alpha_{v0}\alpha_{\ell 0})^{\frac{1}{2}}}; \quad S = \dfrac{2\sigma}{p_0 R_0}; \quad Re = \dfrac{a_* R_0}{4\nu_\ell}; \quad Pe = \dfrac{a_* R_0}{D_\ell}; \\ \alpha_{v0}; \quad \epsilon_0 = \dfrac{\rho_{v0}^0}{\rho_\ell^0}; \quad \gamma; \quad C_{v*} = \dfrac{BT_0}{l} = \dfrac{\gamma - 1}{\gamma}\dfrac{c_{pv} T_0}{l}; \\ C_{\ell *} = \dfrac{c_\ell T_0}{l}; \quad \Delta P_e = \dfrac{p_e - p_0}{p_0}; \quad \left(a_*^2 = \dfrac{p_0}{\rho_\ell^0}\right); \end{array}\right\} \tag{12}$$

which characterize the influence of the external mass forces G, capillary effects S, liquid viscosity (the Reynolds number Re), thermal conductivity (the Péclet number Pe), small relative density ϵ_0 of the vapour, its specific heat ratio γ, specific heats of the phases and the specific heat of vaporization C_{v*}, $C_{\ell *}$, and, finally, of the shock-wave intensity (ΔP_e). Here $D_\ell = \lambda_\ell / \rho_\ell^0 c_\ell$ is the thermal diffusivity of the liquid, the subscript e refers to the parameters of the final equilibrium state (behind the wave).

Under the assumptions made, γ and Pe are constant, and C_{v*} and $C_{\ell *}$ are slowly growing functions of pressure. If pressure differences are not too large, C_{v*} and $C_{\ell *}$ may also be regarded as constants. In the absence of external mass forces the solution of the problem does not depend on G. For not very small bubbles with $R_0 \gtrsim 10^{-3}$ mm, in not very viscous liquids (water, for instance, when $Re \gg 1$) the influence of the capillary effects S and viscosity Re on the wave process in a liquid with bubbles of an insoluble and non-condensable gas is hardly noticeable against the background of heat dissipation (Nigmatulin *et al.* 1974). The results obtained prove that for a liquid with vapour bubbles where the interphase heat and mass transfer plays a more important role, the influence of these effects on the wave process is less significant against the background of heat effect (Zuong Ngok Hai *et al.* 1982).

The calculation results were checked by comparing the steady structure realized after a sufficiently long evolution of non-steady shock waves with the same structure calculated by another method of stationary theory (Zuong Ngok Hai *et al.* 1982). Test calculations for the particular case when mass transfer is absent have been carried out. The solutions obtained were compared with the results of investigations of waves in a liquid with bubbles of an insoluble and non-condensable gas (Gubaidullin *et al.* 1976). The calculation results were also verified by comparison with the solutions to the problem of unsteady thermal conduction around a sphere of constant radius (Carslaw & Jaeger 1959) and the dynamics of a single vapour bubble in a variable pressure field (Nigmatulin *et al.* 1981).

Figure 1 illustrates, to the same scale and under the same conditions, the evolution of a shock wave in water with (*a*) bubbles of an insoluble and non-condensable gas (air) at $T_0 = 293$ K and (*b*) in water boiling at $T_0 = 373$ K with vapour bubbles for the case of steady boundary conditions on $\xi = 0$; $p(0,t) = p_e =$ const. at $L \to +\infty$. The strong influence of the interphase mass transfer on the propagation of pressure waves in vapour–liquid media is evident. In liquids with gas bubbles ($R \sim 1$ mm) the time and distance required for shock waves to achieve their steady structure are several milliseconds and metres, respectively (Gubaidullin *et al.* 1976), whereas these quantities are much smaller in liquids with vapour bubbles. The steady configuration of the waves in figure 1(*b*) is achieved at a distance of about 0.4 m for about 5 ms. In this case the oscillation amplitude decreases, and the wave propagation velocity is practically constant.

The investigation of non-stationary shock waves in different liquids with vapour bubbles shows that the initial stage of propagation of waves with finite intensities is always an oscillatory one. Then the wave acquires a steady configuration which in the case of water at 0.1 MPa and $R_0 \sim 1$ mm (the corresponding values of the basic dimensionless parameters are given in table 1) is monotonic for waves with intensities of $\Delta P_e \lesssim 1$.

The influence of the initial parameters and perturbation parameters of the mixture on the wave evolution has been investigated. It is found that with an increase in the initial void fraction the velocity and wavelength of oscillatory waves decrease. The distance travelled by the waves before their steady propagation is achieved changes as their velocity varies, the time required for the waves to become steady being practically unchanged. With an increase in the initial diameter of the bubbles, other conditions being equal, the oscillation amplitude and the oscillatory wavelengths behind the front grow, the time and distance for reaching steady configurations increase. An increase in the shock-wave intensity leads to an increase in its velocity and an increase in the amplitude and frequency of oscillation of the parameters behind the front. In this case the time taken to reach the steady configuration decreases while the corresponding distance remains practically unchanged. An increase in the initial static pressure p_0 of the mixture leads not only to an increase in the velocity of shock waves with the same dimensionless intensity (ΔP_e) but also to an increase in the lengths and amplitudes of oscillatory waves behind the wave front, as well as in the distance of transition to the steady regime. This is connected

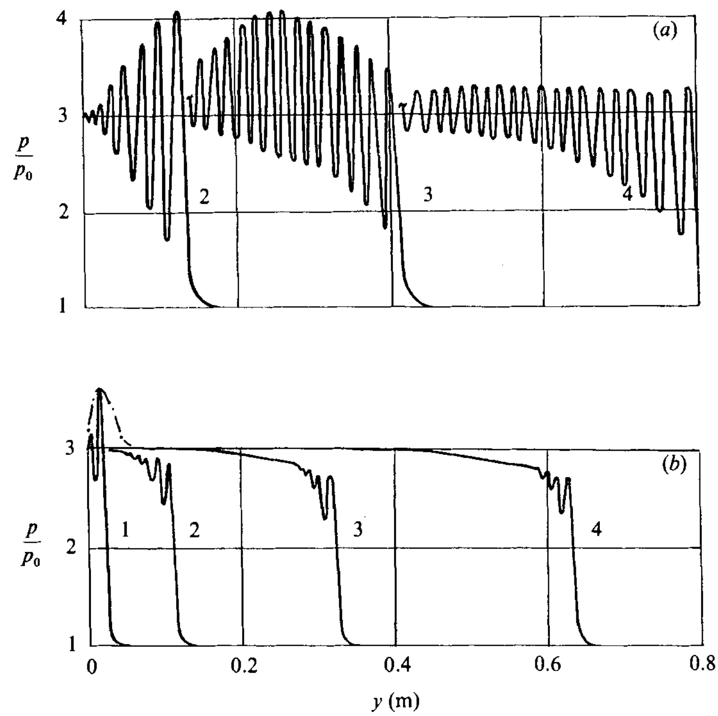

FIGURE 1. Shock-wave evolution in (a) water with bubbles of air and (b) water vapour: $p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\Delta P_e = 2$, $\alpha_{v0} = 0.02$. Curves 1–4 correspond to times $t = 0.2$, 1, 3 and 6 ms; —·—·—, envelope of pressure peaks.

with an increase in the initial mass of the bubbles and with the slowing down of condensation (Zuong Ngok Hai *et al.* 1984).

In waves of moderate intensities ($\Delta P_e \sim 1$–10) the medium parameters vary over a sufficiently wide range. An analysis of the dimensionless equations obtained shows that when $\epsilon_0 = \rho_v^0/\rho_\ell^0 \ll 1$, the effect of ϵ_0 on the wave process can be expressed in a combination with $C_{\ell *}$. In this case the basic similarity criteria in this class of problems are the dimensionless wave intensity ΔP_e and dimensionless numbers and combinations $Pe, C_{\ell *}/\epsilon_0, \gamma, C_{v*}$ which express the influence of the interphase heat and mass transfer (heat dissipation) on the process.

The values of the basic dimensionless numbers and combinations for two-phase single-component systems are listed in table 1.

From table 1 it is seen that the parameters γ and C_{v*} exhibit a slight dependence on medium. For this reason Pe and $C_{\ell *}/\epsilon_0$, together with the wave intensity, are the main similarity criteria in modelling wave processes in these media, the parameter being ϵ_0 easily changed since a change in p_0 contributes considerably to a change in $C_{\ell *}/\epsilon_0$. These similarity criteria allow us to use vapour–liquid media, direct experiments with which are either expensive or impossible. From table 1 it is also clear that the wave picture in a mixture of boiling water with vapour bubbles must resemble that in a mixture of liquid nitrogen with vapour bubbles when the initial pressure in the first mixture increases.

Figure 2(a) illustrates the evolution of a shock wave in liquid nitrogen with vapour bubbles. Comparison with figure 1(b) shows the decisive influence of the ratio ϵ_0 of the densities of the phases and the thermophysical properties of the liquid on the wave

Vapour–liquid medium	p_0 (10^5 Pa)	T_0 (K)	R_0 (mm)	$\epsilon_0 \times 10^3$	$C_{\ell *}$	γ	C_{v*}	$\frac{C_{\ell *}}{\epsilon_0}$	$Pe \times 10^{-5}$
Water	0.5	354	1	0.34	0.51	1.31	0.071	1500	0.44
	1	373	1	0.6	0.70	1.28	0.076	1130	0.61
	5	424	1	2.9	0.86	1.24	0.090	301	1.34
Freon-21	1.85	299	1	6.1	1.40	1.19	0.096	230	1.60
Nitrogen	1	77.35	1	5.7	0.76	1.32	0.086	133	1.89
Freon-12	7	301	1	31.0	2.03	1.18	0.118	65	4.30
	13	326	1	62.6	3.00	1.24	0.140	48	7.23

TABLE 1. The values of the basic dimensionless numbers and combinations for two-phase single-component systems

evolution in vapour–liquid bubbly media. An increase in ϵ_0 and Pe implies an increase in vapour mass in bubbles and a slowing of the interphase heat and mass transfer. This leads to an increase in the time and distance of wave transition to the limiting steady configuration, to an increase in wave thickness and to a stronger tendency towards oscillations in a wave, which are characteristic of wave propagation in a liquid with bubbles of an insoluble and non-condensable gas (figure 1*a*). The non-steady wave in liquid nitrogen with vapour bubbles presented in figure 2(*a*) exhibits a pronounced oscillatory structure. The wave acquires a monotonic steady structure at a distance of the order of 10 m, the wave thickness reaching the order of 3 m (see figure 2*b*).

From figure 2(*a*) it is also seen that at the initial non-steady stage of wave propagation the pressure in the wave can exceed the pressure behind the wave (enhancement effect). In this case a maximum enhancement is not observed immediately after the wave initiation and not necessarily in the first peak of pressure pulsation. The dash-dotted line in figure 2(*a*) is the envelope of the pressure peaks. An increase in pressure at the non-steady stage of shock-wave propagation in these media becomes more pronounced with an increase in the initial pressure in the system and in shock intensity. For waves with an intensity of $\Delta P_e = 4$ ($p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\alpha_{v0} = 0.05$, $C_{\ell *}/\epsilon_0 = 1130$, $Pe = 61 \times 10^3$) in boiling water with vapour bubbles at the non-steady stage, the estimated pressure increase is

$$\Delta P_{\max} = (p_{\max} - p_0)/p_0 = 7 \approx 2\Delta P_e$$

(Zuong Ngok Hai *et al.* 1984). A more significant enhancement of waves of higher intensity was observed experimentally by Borisov *et al.* (1977, 1982), bubble fragmentation and vapour-condensation acceleration contributing to this. In the absence of bubble fragmentation the maximum enhancement will occur at $Pe = 0$, which provides the Rayleigh regime of collapse ($p_v = \text{const}$). According to the solution (Nigmatulin *et al.* 1982)

$$p_{\max} = p_e\left(1 + \frac{p_e - p_0}{p_e}\frac{\rho_\ell^0 a_\ell^2}{p_e}\frac{\alpha_{v0}}{\alpha_{\ell 0}}\right)^{\frac{1}{2}}.$$

For the above values of the parameters the pressure increase reaches $\sim 13p_e$, i.e. $\Delta P_{\max} \approx 16\Delta P_e$.

If nucleation, fragmentation and coagulation of the bubbles are absent, the length of the wave transition zone (distance of wave transition to a steady configuration)

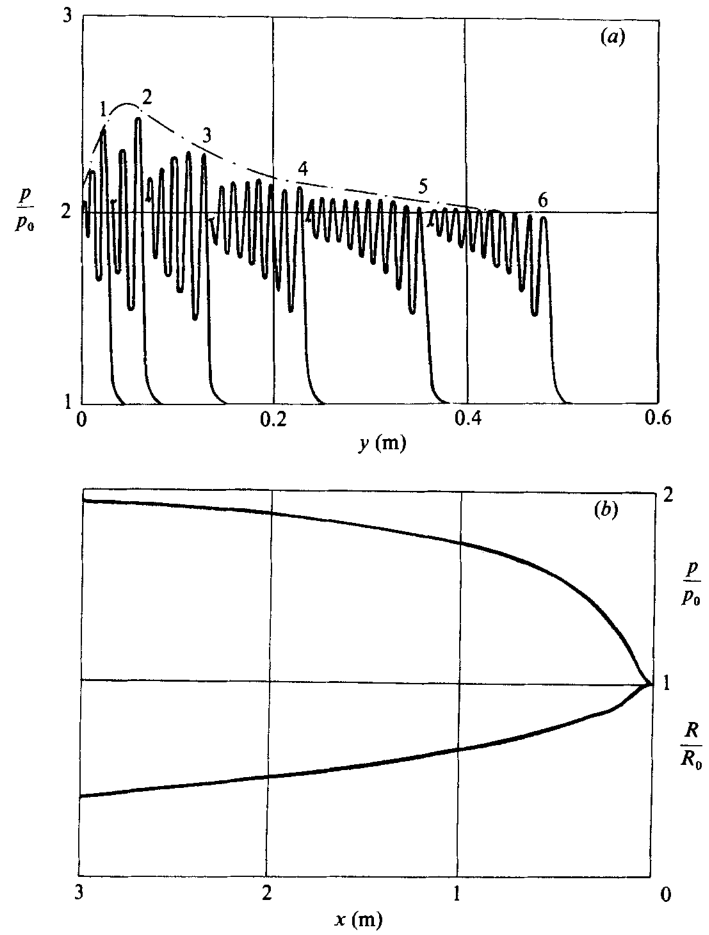

FIGURE 2. (*a*) Shock-wave evolution in liquid nitrogen with vapour bubbles: $p_0 = 0.1$ MPa, $R_0 =$ 1 mm, $\Delta P_e = 1$, $\alpha_{v0} = 0.05$. Curves 1–6 correspond to times $t = 0.49$; 0.90; 1.80; 3.15; 4.95 and 6.75 ms; —·—·—, envelope of pressure peaks. (*b*) Structure of steady shock wave, $\Delta P_e = 1$, in liquid nitrogen with vapour bubbles: $p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\alpha_{v0} = 0.05$.

strongly depends on the mixture structure and changes rather weakly with the wave intensity change. An analysis of the results obtained and of the dimensionless quantities has shown that the transition-zone length is expressed by the following formula:

$$L_{st} = \frac{R_0}{(\alpha_{v0}\,\alpha_{\ell 0}(\Delta P_e)^{\frac{1}{2}})^{\frac{1}{2}}}\, f_{st}\left(Pe, \frac{C_{\ell *}}{\epsilon_0}, \gamma, C_{v*}\right). \tag{13}$$

In the case of a vapour–water mixture when $Pe = 61 \times 10^3, C_{\ell *}/\epsilon_0 = 1130$, dependence (13) quite well approximates the results of calculations at $f_{st} \approx 55$, viz. at $\Delta P_e \sim 1$, $R_0 \sim 1$ mm, $\alpha_{v0} \sim 10^{-2}$, L_{st} is of the order of 0.3 m. For some other cases the change in the wave transition-zone length L_{st} which depends on interphase heat and mass transfer, is presented in table 2. Since the parameters γ and C_{v*} vary but insignificantly with medium, they can be omitted in the functional dependence (13). The results listed in table 2 were calculated for mixtures of $\alpha_{v0} = 0.05$ and waves with an intensity of $\Delta P_e = 1$.

From table 2 one can see a strong influence of interphase heat and mass transfer,

Vapour–liquid medium	p_0 (10^5 Pa)	R_0 (mm)	$\frac{C_{\ell *}}{\epsilon_0}$	$Pe \times 10^{-5}$	L_{st} (m)	f_{st}	$\Delta X \times 10^2$ (m)	f_w
Water	0.5	1	1500	0.44	0.12	26	1	2.2
	1	0.1	1130	0.06	0.005	11	0.1	2.2
	1	1	1130	0.61	0.25	55	2	4.5
	1	10	1130	6.10	7	154	40	9
	5	1	301	1.34	6	1320	15	35
Nitrogen	1	1	133	1.89	10	22×10^3	50	120

TABLE 2. The wave transition-zone length and the steady shock-wave front thickness for two-phase single component systems.

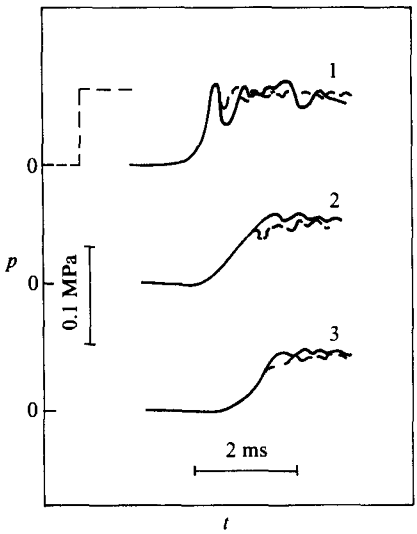

FIGURE 3. Comparison of calculated shock-wave evolution in a vapour–liquid mixture of freon-21 with experimental data by Nakoryakov *et al.* (1984). ---, theory; —, experiment. Curves 1–3 correspond to cross-sections $y = 0.11$; 0.36 and 0.46 m.

mainly via the parameters Pe and $C_{\ell *}/\epsilon_0$, on the length of the wave transition zone. The transition time t_{st} is related to the length of the wave transition zone as follows:

$$t_{st} = \int_0^{L_{st}} \frac{d\xi}{U(\xi)} \approx \frac{L_{st}}{U_0}, \quad U_0^2 = \frac{p_e - p_0}{\alpha_{\ell 0}\alpha_{v0}\rho_\ell^0}. \tag{14}$$

The zone of pressure increase in a wave under normal conditions ($p_0 \sim 0.1$ MPa) is very narrow for many mixtures. This zone becomes wider with a decrease in the intensity of the interphase heat and mass transfer and with a wave velocity increase. When the initial pressure in water with vapour bubbles is increased from 0.1 to 0.5 MPa, the width of the zone, in which enhancement of waves with an intensity of $\Delta P_e = 1$ occurs, changes from 1 cm to 10 cm. In liquid nitrogen with vapour bubbles the width of the zone is about 0.4 m (figure 2*a*). In the general case the width of this zone is a complex function of the parameters determined by nonlinear and dissipative effects in the medium, $\Delta P_e, Pe, C_{\ell *}/\epsilon_0, \gamma, C_{v*}$; see (12).

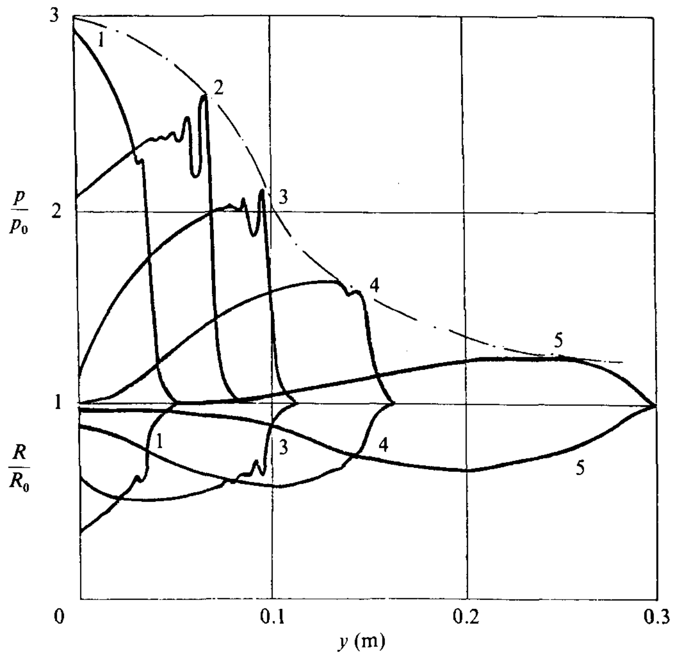

FIGURE 4. Evolution of delta-pulse in water with vapour bubbles: $p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\Delta P_{max} = 2$, $\alpha_{v0} = 0.05$, $t_0 = t_2 = 2t_1 = 2$ ms. Curves 1–5 correspond to times $t = 1$; 1.5; 2; 3 and 6.5 ms; —·—·—, envelope of pressure peaks.

Figure 3 presents a comparison of the calculated pressure profiles in shock waves in liquid freon-21 with vapour bubbles (dashed curves) with the experimental profiles in cross-sections of the shock tube positioned at different distances from the free surface of the working section (solid curves) (Nakoryakov *et al.* 1984) with $p_0 = 0.1$ 85 MPa and $\Delta P_e = 0.4$. The average initial radius of the bubbles is 1.2 mm, and the initial vapour content is 0.01; in this case the initial state created is similar to the condition of a monodisperse and equilibrium mixture. In this experiment the wave failed to reach the steady regime of propagation owing to the limited length of the experimental section. It is clear that the pressure pulsations observed in the experiment are fairly well described by the numerical solutions of the system of differential equations (2)–(11). The results of the measurements cover a time interval till the moment when the shock wave reflected from the bottom of the low-pressure chamber reaches the transducers.

In spite of some spread in diameter of the bubbles and their non-sphericity observed in the experiments, calculations within the framework of the model of the monodisperse mixture with spherical bubbles are in satisfactory agreement with experiment as far as the amplitude, frequency and number of oscillations in the wave are concerned.

The evolution of a pulse disturbance of finite duration depends not only on the above parameters but also on the type of pulse and its duration t_0 or on the wavelength $L_0 = U_0 t_0$. Thus, a new parameter $d = L_0(\alpha_{\ell 0}\alpha_{v0})^{\frac{1}{2}}/R$ reflecting the influence of the dispersity of a mixture of a liquid with vapour bubbles is added to the list of basic parameters. Depending on this parameter, the other parameters $\Delta P_{max} = (p_{max} - p_0)/p_0$, Pe, $C_{\ell *}/\epsilon_0$ etc. being fixed, the delta-pulse can be transformed into solitary waves (solitons) or into an oscillation train (wave packet) (Nakoryakov

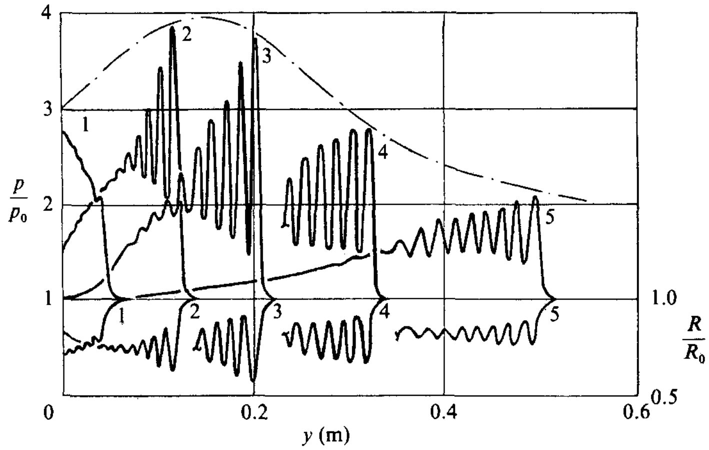

FIGURE 5. Evolution of delta-pulse in liquid nitrogen with vapour bubbles: $p_0 = 0.1$ MPa, $R_0 = 1$ mm $\Delta P_e = 2$, $\alpha_{v0} = 0.05$, $t_0 = t_2 = 2t_1 = 2$ ms. Curves 1–5 correspond to times $t = 0.9$; 1.8; 2.7; 4.0 and 6.4 ms; —·—·—, envelope of pressure peaks.

et al. 1984). For low dissipation in the absence of phase transition one can find a d_{cr} (Berezin & Karpman 1966) such that at $d > d_{cr}$ the perturbation is transformed into solitary waves, and at $d < d_{cr}$ into an oscillation train. In the case of strong dissipation, significant nonlinearity and in the presence of phase transition, wave evolution strongly depends not only on d but on these processes as well. However, at fixed ΔP_{max}, Pe, $C_{\ell *}/\epsilon_0$ one can determine the value of d_{cr} experimentally or numerically even in these cases. The condition $d \rightarrow +\infty$ or $L_0 \rightarrow +\infty$ means a transition from a pulse disturbance of finite duration to a shock wave. Figure 4 illustrates a typical wave picture of the delta-pulse evolution. The profiles of pressure and bubble radius in a mixture of boiling water with vapour bubbles at different times are presented. The dash-dotted line is an envelope of pressure peaks at different times. In this case $Pe = 61 \times 10^3$; $C_{\ell *}/\epsilon_0 = 1130$; $d \approx 67$. It can be seen that the pulse evolution in vapour–liquid media is characterized by a stronger damping than in gas–liquid media owing to the significant influence of the interphase heat and mass transfer. At a distance of the order of 0.1 m the pulse amplitude is only half the initial value, whereas at a distance of the order of 0.5 m the pulse is practically not observed. In this case the perturbation is oscillatory at the initial stage but owing to a strong influence of the interphase heat and mass transfer it goes over into a solitary wave and damps quickly.

An estimation shows that in this case the time of bubble collapse due to vapour condensation ($\Delta t_c \sim 10^{-1}$s) considerably exceeds the pulse action time ($\Delta t \sim 10^{-3}$s). For this reason, when the bubbles contract owing to an increase in pressure in the liquid, their mass under the conditions of figure 4 varies insignificantly, and the bubbles begin to expand in the zone of decreasing pressure. Condensation gives way to evaporation since the bubble surface temperature drops rapidly as the pressure decreases ($T_\sigma = T_s(p_v)$), whereas the near-wall liquid is still warm, and the heat flux from the liquid to the interface changes its sign. Though the bubble equilibrium defined by (11) is unstable, the characteristic time of the development of instability is much longer than the pulse action time. As a result, instability fails to manifest itself for this period of time.

This explains the fact that the bubbles return to practically their initial

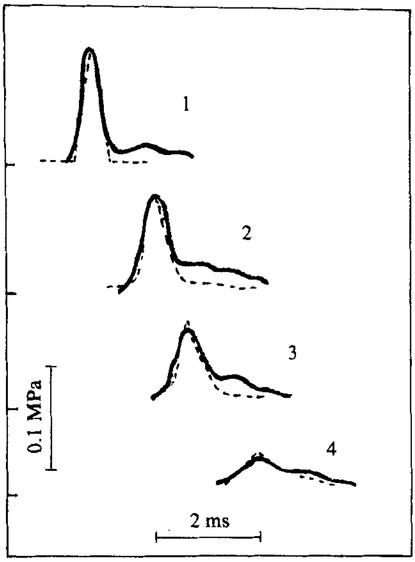

FIGURE 6. Comparison of calculated evolution of delta-pulse in water with vapour bubbles with experimental data by Pokusaev (1979); $p_0 = 0.5$ MPa, $R_0 = 1.4$ mm, $\Delta P_{max} = 0.2$, $\alpha_{v0} = 0.015$. ---, theory; —, experiment. Curves 1–4 correspond to the following coordinates of the shock-tube sections measured from the free surface of the working section; $y = 0$; 0.05; 0.15 and 0.23 m.

dimensions after the shock pulse has passed. The damping of the delta-pulse strongly depends on its initial duration. More intense and longer pulses exhibit a lower damping. During the evolution of the delta-pulse as well as during the propagation of non-steady shock waves the effect of pressure increase in the mixture, as compared to the initiating pressure, is observed.

Figure 5 illustrates the evolution of the delta-pulse in boiling liquid nitrogen containing bubbles with saturated vapour. A pronounced pulsatory character of wave propagation is seen. Calculations show that in this medium the propagation of the delta-pulse may be of a pulsatory character even at $\Delta P_{max} = 0.4$, which means that the interphase heat and mass transfer in it is less intense than in water. The evolution patern in figure 5 is similar to the soliton regime of pulse propagation in gas–liquid media. In this case at $t = 1.8$ and 2.7 ms (curves 2, 3 in figure 5) enhancement of the initial pulse is observed.

Of particular interest is a comparison of theoretical and experimental pressure profiles in short shock waves, the initial duration of which has a strong effect on the evolution process. Figure 6 compares the calculated evolution of a delta-pulse in boiling water containing vapour bubbles (dashed curves) with experimental data (Pokusaev 1979) (solid curves). One can see good agreement between the experimental and calculated pressure profiles.

Thus, a comparison of a set of experimental pressure profiles with calculated ones showns that the theory is applicable as an adequate description of the evolution of long and short shock waves on condition that they are not very strong and there is no intense fragmentation of bubbles.

4. The structure of steady shock waves

The steady shock wave is realized as a result of a steady or sufficiently prolonged action on the mixture. To study steady shock waves, one may use the above system of differential equations (2)–(10). Depending on different factors (basic parameters α_{v0}, ΔP_e, Pe, $C_{\ell *}/\epsilon_0$) a steady wave can form at distances from several centimetres to dozens of metres. The relaxation time can amount to dozens of milliseconds, and the wave thickness to several metres. In the general case the evolution of shock waves also depends on γ, C_{v*}, S, Re, etc., see (12). However, the steady configuration of shock waves can be readily found in the system of Eulerian coordinates connected with the wave in which the latter is motionless. In this coordinate system differential equations (1), (5) and (7) simplify to

$$\left.\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\rho v)=0;\quad \frac{\mathrm{d}}{\mathrm{d}x}(\rho_v v)=4\pi R^2 nj;\quad \frac{\mathrm{d}}{\mathrm{d}x}(nv)=0;\\
&\frac{\mathrm{d}}{\mathrm{d}x}(\rho v^2+p)=0;\quad \rho_\ell^0 c_\ell\left(v\frac{\partial T_\ell}{\partial x}+w_{\ell\sigma}\frac{R^2}{r^2}\frac{\partial T_\ell}{\partial r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(\lambda_\ell r^2\frac{\partial T_\ell}{\partial r}\right)\\
&Rv\frac{\mathrm{d}w_{\ell\sigma}}{\mathrm{d}x}+1.5\,w_{\ell\sigma}^2+\frac{4\nu_\ell}{R}w_{\ell\sigma}=\frac{1}{\rho_\ell^0}\left(p_v-p_\ell-\frac{2\sigma}{R}\right);\\
&v\frac{\mathrm{d}R}{\mathrm{d}x}=w_{\ell\sigma}+\frac{j}{\rho_\ell^0}=w_{v\sigma}+\frac{j}{\rho_v^0};\quad q_{v\sigma}=\tfrac{1}{3}R\left[\frac{c_{pv}T_v}{l}\left(1-\frac{\rho_v^0}{\rho_\ell^0}\right)-1\right]v\frac{\mathrm{d}p}{\mathrm{d}x};
\end{aligned}\right\}\qquad(15)$$

the other equations and relationships being unchanged. In this case the equation of conservation of the mass of the mixture is used in (1) instead of the equations expressing a change in the mass of the liquid phase. An investigation of the structure of steady shock waves in a mixture of a liquid with vapour bubbles implies seeking the solution to (15) with the following boundary conditions:

$$\left.\begin{aligned}
&\left.\begin{aligned}&p_\ell=p_0;\quad p_v=p_0+\frac{2\sigma}{R_0};\quad R=R_0;\quad T_\ell=T_v=T_0;\\ &v=v_0;\quad w_{\ell\sigma}=w_{v\sigma}=0.\end{aligned}\right\}\ (x\to-\infty)\\
&p=p_e=\text{const.}\qquad\qquad (x\to+\infty).
\end{aligned}\right\}\qquad(16)$$

The existence of this solution means the existence of steady waves. During the propagation of shock waves of finite intensity in a liquid mixture with vapour bubbles the temperature of the medium behind the wave increases, mainly owing to the release of heat due to vapour condensation in the bubbles. Under normal conditions, $p\sim 0.1$ MPa, for most substances at small void fractions $\alpha_v\sim 10^{-2}$, this increase is low:

$$\Delta T\sim\frac{\alpha_v\rho_v^0 l}{\alpha_\ell\rho_\ell^0 c_\ell}\sim 10^{-2}\,\mathrm{K}.$$

The temperature also increases owing to the transformation of the kinetic energy of small-scale motion around the bubbles into heat. The kinetic energy of small-scale motion is given by the expression (Nigmatulin 1978, 1979)

$$k_r=2\pi R^3 n w_{\ell\sigma}^2=\tfrac{3}{2}\alpha_{v0}\,w_{\ell\sigma}^2.$$

The kinetic energy of small-scale motion is maximum in the case of bubble collapse in the inertial Rayleigh regime, and in this case

$$k_r^{(\mathrm{R})} \approx \frac{\alpha_{\mathrm{v0}}(p_{\mathrm{e}}-p_0)}{\rho_\ell^0}.$$

Even if this energy is completely transformed into heat, an increase in the medium temperature is small:

$$\Delta T \sim \frac{k_r^{(\mathrm{R})}}{c_\ell} = \alpha_{\mathrm{v0}}\frac{p_{\mathrm{e}}-p_0}{\rho_\ell^0 c_\ell} \sim 10^{-3}\ \mathrm{K}.$$

For this reason, the average medium temperature behind the shock-wave front does not in practice differ from the initial T_0, i.e. the medium in front of and behind the wave has the same temperature, but different pressures: a two-phase vapour–liquid mixture in front of the wave and single-phase subcooled liquid behind the wave.

The system of differential equations (15) has first integrals

$$\rho v = \rho_0 v_0; \quad nv = n_0 v_0; \quad \rho v^2 + p = \rho_0 v_0^2 + p_0; \tag{17}$$

from which one can obtain the structure of steady shock waves in parametric form:

$$\left.\begin{aligned} &p = p_0 + (p_{\mathrm{e}}-p_0)\left[1-\left(\frac{R}{R_0}\right)^3\right]; \quad v = v_0\left[\alpha_{\ell 0}+\alpha_{\mathrm{v0}}\left(\frac{R}{R_0}\right)^3\right];\\ &\frac{1}{\rho} = \frac{\alpha_{\ell 0}}{\rho_0}\left[1+\frac{\alpha_{\mathrm{v0}}}{\alpha_{\ell 0}}\left(\frac{R}{R_0}\right)^3\right]; \quad v_0^2 = \frac{p_{\mathrm{e}}-p_0}{\alpha_{\mathrm{v0}}\rho_0} \approx U_0^2; \quad v_{\mathrm{e}} = \alpha_{\ell 0} v_0 \end{aligned}\right\} \tag{18}$$

From the first expression of (18) it follows that $p(x) \leqslant p_{\mathrm{e}}$. The estimation obtained shows that, unlike unsteady waves, the pressure in steady shock waves in a liquid with vapour bubbles cannot exceed the pressure behind the wave p_{e}, i.e. the effect of wave enhancement cannot occur in a steady wave. An insignificant enhancement of steady shock waves can take place in an incompressible liquid with gas bubbles of constant mass because, as a result of pulsations, the bubbles may pass over the final equilibrium state (Zuong Ngok Hai *et al.* 1982):

$$p(x) = p_0 + p_{\mathrm{e}}\left[1-\left(\frac{R}{R_0}\right)^3\right] < p_{\mathrm{e}} + p_0.$$

It is seen that in this case the pressure within a wave cannot exceed the pressure behind it by an amount equal to the initial pressure. This enhancement ($p > p_{\mathrm{e}}$) becomes more significant in unsteady waves in the region of their initiation.

From the first integrals of conservation of mass and momentum of the mixture it also follows that

$$p(x) = p_0 + \rho_0 v_0^2\left(1-\frac{\rho_0}{\rho}\right), \tag{19}$$

i.e. $p = f(\rho)$. This relation is the consequence of the hydrodynamic equations and means that the mixture pressure within steady waves is a function of the mixture density only. The equation of state of the medium obtained from the third relation in (18) and from the Rayleigh–Plesset equation is of the form

$$p(x) = f\left(\rho, \frac{\mathrm{d}\rho}{\mathrm{d}x}, \frac{\mathrm{d}^2\rho}{\mathrm{d}x^2}, p_{\mathrm{v}}\right). \tag{20}$$

Equations (19) and (20), under certain boundary conditions, completely determine the steady wave structure. Though the parameters do not vary with time at each point in space, all the parameters of a fixed material particle, including a test bubble, are time-dependent. This non-stationarity has a marked effect on heat and mass transfer. In the general case the solution to these two equations can be obtained only numerically owing to their complex nonlinearity (and because it is necessary to solve the non-stationary nonlinear heat problem in the liquid). However, in some limiting cases this solution can be obtained by a simpler method. When the pressure within the bubbles is constant, $p_v = \text{const}$ (i.e. when they collapse in the inertial Rayleigh regime), which is realized if the liquid thermal conductivity is sufficiently high ($\lambda_\ell \to +\infty$), the solution of this system can be expressed in the form of the integral (Nigmatulin *et al.* 1982)

$$x = \int_0^R \frac{\alpha_{\ell 0} v_0 \left[1 + \dfrac{\alpha_{v0}}{\alpha_{\ell 0}} \left(\dfrac{R}{R_0}\right)^3\right]}{\left\{\dfrac{2}{3} \dfrac{p_e - p_0}{\rho_\ell^0} \left[\left(\dfrac{R_0}{R}\right)^3 - 1\right]\right\}^{\frac{1}{2}}} \mathrm{d}R. \tag{21}$$

In this limiting case the structure of steady shock waves is entirely determined by (18) and (21). Some other cases could also be considered when simple dependences of p_v on other parameters occur.

To investigate the asymptotic behaviour of the solution of (15) in the neighbourhood of the initial equilibrium state, these equations need to be linearizeed with respect to the values of the parameters at the point 0. Their solution is sought in the form of an exponential damped as $x \to -\infty$

$$\Phi = \Phi_0 + A_\Phi \exp(hx); \quad \mathrm{Re}\, h > 0. \tag{22}$$

In this case Φ is any parameter to be sought. After linearization we obtain from (15) a system of linear homogeneous equations in the amplitudes A_Φ. The condition of the existence of a non-trivial solution to this system leads to an algebraic equation of sixth order with respect to $H = (hv_0)^{\frac{1}{2}}$. It has been proved (Zuong Ngok Hai *et al.* 1982) that at $\Delta P_e > 0$ (shock wave) the desired solution exists and is unique. This proves that in the presence of interphase heat and mass transfer there exists a unique solution of (22) for all $\Delta P_e > 0$. The states in front of ($x = -\infty$) and behind the shock wave ($x = +\infty$) are equilibrium ones, which is why an analysis of the asymptotic behaviour at $x \to -\infty$ is necessary to 'get away' from the singular point. The integral curve connecting these two states and determining the structure of the steady shock wave is found by the numerical continuation of the linear solution obtained in the neighbourhood of the singular point.

The integral curves of the system of basic equations (15) allow displacement along the x-axis. Therefore, for $x = 0$ we fix some value of the dimensionless radius of the bubble $R_* = R/R_0$, which must be taken sufficiently close to 1 in order that the linear solution holds in the region $x < 0$. From the system of linearized equations (15) one can, given the amplitude of the perturbation of the bubble radius and the value h of the root of the dispersion equation, determine the values of the remaining amplitudes at $x = 0$. These quantities determine the boundary conditions for numerical solution of the nonlinear problem in the region $x > 0$. An analysis of the second singular point, which corresponds to the equilibrium behind the wave, can be made in a similar manner.

Figure 7 presents the calculated structures of steady shock waves (profiles of

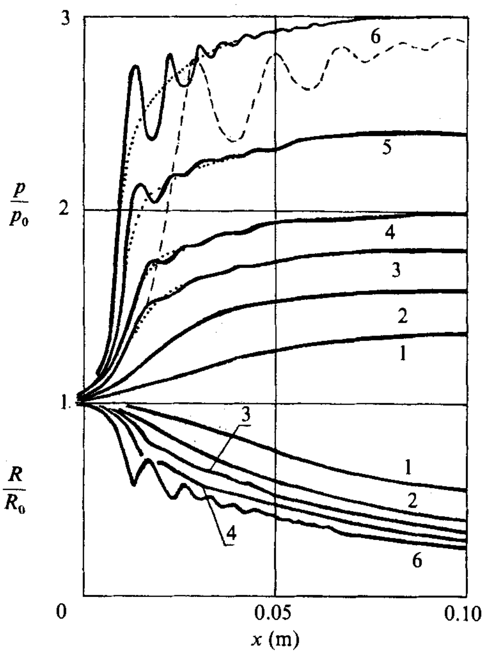

FIGURE 7. Structure of steady shock waves in water with vapour bubbles: $p_0 = 0.1$ MPa, $R_0 = 1$ mm. Curves 1–6 correspond to different wave intensities: $\Delta P_e = 0.4$; 0.6; 0.8; 1.0; 1.4 and 2.0, — $\alpha_{v0} = 0.05$; ---, $\alpha_{v0} = 0.01$; $\Delta P_e = 2$, ..., averaged pressure profiles.

pressure and bubbles radius) in boiling water with vapour bubbles. For weak waves with a monotonic structure the phase pressures are equal, $p_v = p_\ell$. In the case of oscillatory waves the pressure in the vapour phase pulsates synchronously with the pressure in the liquid but with an appreciably greater amplitude (figure 8). In this case the dimensionless heat flux expressed through the Nusselt number

$$Nu_\ell = \frac{2R\,(\partial T_\ell/\partial r)_{r=R}}{(T_0 - T_v)}$$

(Zuong Ngok Hai & Khabeev 1983) and the kinetic energy of small-scale motion of the liquid around bubbles $k_r = 1.5 w_{\ell\sigma}^2 \alpha_v (1 - \alpha_v^{\frac{1}{3}})$ are of an oscillatory nature. In contrast to shock waves in a liquid with bubbles of an insoluble and non-condensable gas (Nigmatulin & Shagapov 1974) when steady waves with an intensity of $\Delta P_e \gtrsim \gamma - 1$ have an oscillatory structure, the existence of mass transfer significantly enhances the tendency of the waves to have a monotonic structure and widens the range of intensities for which such a structure is realized. In the case presented in figure 7 the dimensionless critical intensity of the waves ΔP_{cr} is approximately equal to 1: steady shock waves with an intensity of $\Delta P_e > \Delta P_{cr}$ have an oscillatory structure and a steady shock wave with an intensity of less than ΔP_{cr} has a monotonic structure. It is obvious that in the case of vapour bubbles this dimensionless critical intensityof waves strongly depends on the degree to which the vapour phase is dispersed. Varying the bubble initial radius R_0 corresponds to varying the dimensionless Peclet number *Pe*. In the general case this dimensionless critical intensity of waves ΔP_{cr}, as well as d_{cr}, is a function of the above basic parameters *Pe*, $C_{\ell *}/\epsilon_0$, etc. Figure 7 also illustrates the effect of the initial vapour volume fraction α_{v0} on the structure of

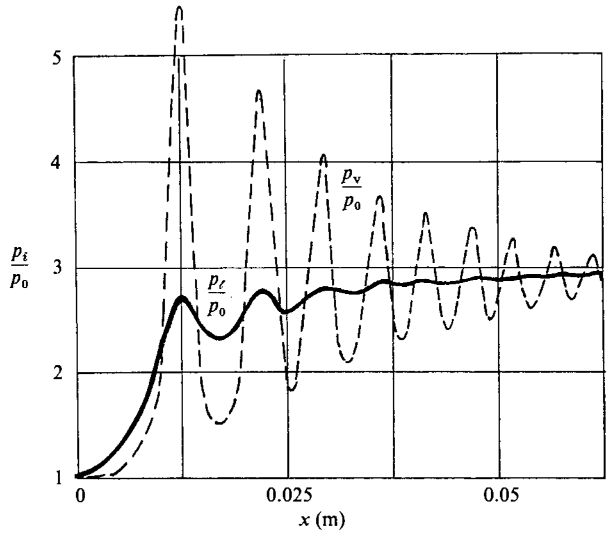

FIGURE 8. Pressure profiles for phases in steady shock waves in water with vapour bubbles: $p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\Delta P_e = 2$, $\alpha_{v0} = 0.05$. —, pressure profile in liquid; ----, pressure profile in vapour.

steady shock waves. The dotted curve is a profile of the pressure in the wave with an intensity of $\Delta P_e = 2$ at $\alpha_{v0} = 0.01$ and other similar parameters. Comparison with the curve $\alpha_{v0} = 0.05$ shows that a decrease in the bubble volume fraction leads to a significant increase in the wave thickness. (In this case the amplitude and number of oscillations do not change.) An investigation of the equations obtained, (15), shows that this is mainly due to an increase in the velocity of the shock wave which is inversely proportional to $\alpha_{v0}^{\frac{1}{2}}$:

$$v = U_0[1+O(\alpha_{v0})] \approx \left(\frac{p_0 \Delta P_e}{\alpha_{\ell 0}\alpha_{v0}\rho_\ell^0}\right)^{\frac{1}{2}}.$$

Subject to minor changes in the medium velocity v in the wave, (18), it follows, all other things being equal, that the configurations of steady shock waves in a liquid with vapour bubbles are spatially similar to each other, the similarity coefficient being equal to the root of the ratio of the void fractions. In the above two cases this coefficient is equal to 5. Calculation results presented in figure 7 prove this.

Let the thickness of such waves be measured from the point where the medium pressure differs from the initial equilibrium pressure p_0 by a certain small amount, for example when $|p-p_0|/|p_e-p_0| = 3\times10^{-2}$, up to the point where it asymptotically approaches the final pressure p_e to differ from the latter by a small value, e.g. $|p_e-p|/|p_e-p_0| = 3\times10^{-2}$. Then from figure 7 one can determine the dependence of the thicknesses of steady shock waves on the nonlinearity parameter ΔP_e, with other parameters $\alpha_{v0} = 0.05$, $Pe = 61\times10^3$, $C_{\ell *}/\epsilon_0 = 1130$ being fixed, which corresponds to the previously mentioned mixture of boiling water with vapour bubbles. This dependence is presented in figure 9 (curve 1). In this case the above method of determining the shock-wave thickness is adopted. From figure 9 it is seen that the thickness of steady (both monotonic and oscillatory) shock waves in a liquid with vapour bubbles nonlinearly depends on the wave intensity ΔP_e.

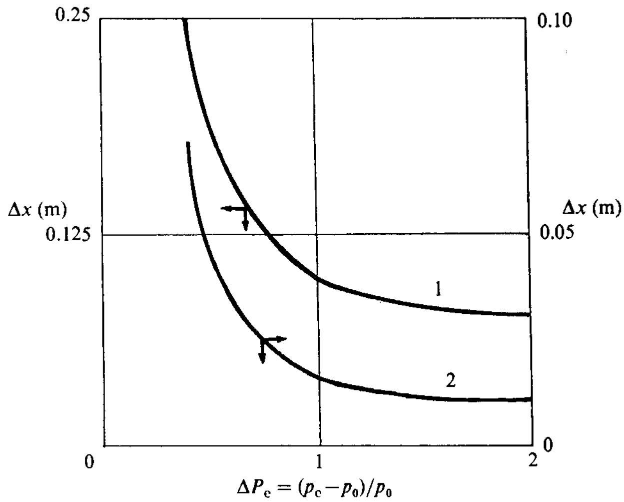

FIGURE 9. Thickness of steady shock waves (curve 1) and thickness of their fronts (curve 2) as a function of intensity ΔP_e: $p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\alpha_{v0} = 0.05$ (see figure 7).

For waves with a monotonic structure, the thickness of the shock-wave front can be determined in a different way, namely

$$\Delta x = \Delta P_e \Big/ \left(\frac{\partial p}{\partial x}\right)_{max} \quad \text{or } \Delta t = \Delta P_e \Big/ \left(\frac{\partial p}{\partial t}\right)_{max} \tag{23}$$

(curve 2, figure 9).

If the wave intensity increases the front thickness decreases. The minimum wave-front thickness is realized when bubbles collapse in the Rayleigh regime, for which $p_v =$ const or $\Delta P_e \to +\infty$, i.e. the thickness of the shock-wave front in a liquid with vapour bubbles asymptotically tends to zero when the wave intensity increases to infinity. The behaviour of the curves in figure 9 shows that with an intensity increase the thickness of the wave front increases. From curve 2 in figure 9 one can see that in the vapour–water mixture at $\alpha_{v0} = 0.05$, $Pe = 61 \times 10^3$, $C_{\ell *}/\epsilon_0 = 1130$ a wave with an intensity of $\Delta P_e = 0.2$ can have a front thickness of the order of 0.2 m whereas the thickness of the whole wave is of the order of 1 m (curve 1 in figure 9). In this case the length of the zone can reach several metres. A decrease in the wave intensity leads to an increase in the wave thickness and transition-zone length. This, in turn, leads to the fact that for weak waves, owing to the limited length of the experimental set-up (about 2–3 m), the pressure profiles thus obtained correspond mainly to non-steady waves.

An analysis of the results obtained and of the dimensionless quantities shows that in order to determine the thickness of the steady shock-wave front (curve 2 in figure 9), when $\alpha_v \gg \alpha_c$, one may use the following expression:

$$\Delta x = \frac{R_0}{(\alpha_{\ell 0}\, \alpha_{v0}\, \Delta P_e^3)^{\frac{1}{2}}} f_w \left(Pe, \frac{C_{\ell *}}{\epsilon_0}, \gamma, C_{v*}\right). \tag{24}$$

In the case presented in figure 9 (curve 2) this expression approximate the calculated data at $f_w \approx 4.5$ quite well. It is shown below that at $\alpha_v \lesssim \alpha_c$ it is necessary to take account of the compressibility of the carrying liquid. Besides, as mentioned

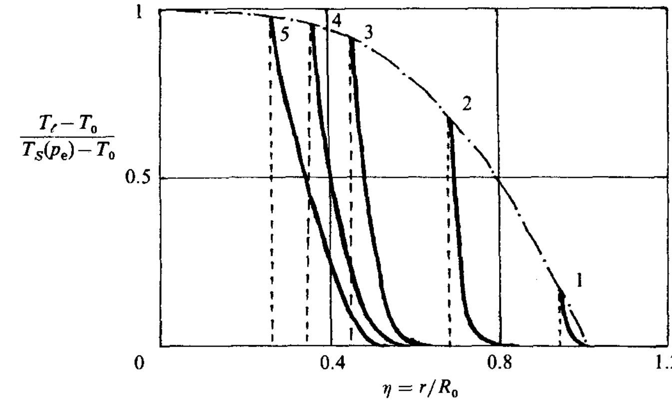

FIGURE 10. Evolution of the thermal boundary layer in liquid around a bubble in the wave $p_0 = 0.1$ MPa, $R_0 = 1$ mm, $\Delta P_e = 0.04$, $\alpha_{v0} = 0.05$ (see curve 1 in figure 7). Curves 1–5 correspond to $X = 0.015; 0.05; 0.115; 0.18$ and 0.245 m. ----, bubble boundary; —·—·—, envelope of temperature peaks in the liquid around a bubble.

above, the thickness of shock waves strongly depends on the dissipation parameters. For example, in nitrogen for waves with an intensity of $\Delta P_e \sim 1$ (figure 2*b*) the thickness of the steady front can reach several dozens of centimetres. And in the vapour–water mixture in the case $\alpha_{v0} = 0.05$, $Pe = 6.1 \times 10^3$, $C_{\ell *}/\epsilon = 1130$ (which corresponds to $R_0 = 0.1$ mm) for shock waves with an intensity of $\Delta P_e \sim 1$ this value is only of the order of several mm. The dependence of the thickness of the steady shock-wave front (determined by (24)) on the dissipation parameters for some cases is presented in table 2. From the table it is seen that dissipative processes have a strong effect on the formation of the structure of steady shock waves in a liquid with vapour bubbles.

An investigation of the heat in the liquid around the bubble plays an important role in studying wave processes in a mixture of liquid with vapour bubbles. Figure 10 illustrates the distribution of temperature in the liquid around the bubble in the wave, $\Delta P_e = 0.4$, at different distances x within the wave (curve 1 in figure 7). Dashed lines show bubble boundaries, and the dash-dotted curve is the envelope of the temperature peaks. It can be seen that the simplifying assumption frequently employed to calculate the dynamics of vapour bubbles, namely, that the thermal boundary layer in the liquid is thin, becomes invalid in the case of bubble collapse when the bubble radius reaches values of $R/R_0 \lesssim 0.4$.

Figure 11 shows that the steady structure of shock waves in water with vapour bubbles depends on the degree to which the vapour phase is dispersed, other things being equal. It is seen that an increase in the degree to which the vapour phase is dispersed (a decrease in the initial radius of bubbles and conservation of the void fraction in the mixture) causing an increase in the interphase heat and mass transfer role due to the specific interphase surface growth, leads to the evolution of shock waves from an oscillatory to a monotonic structure which is realized during bubble inertial collapse. In this case the characteristic wave thickness significantly decreases (about 30 times). At $x \rightarrow -\infty$ the Nusselt number tends to a finite value:

$$Nu_\ell \rightarrow 2\left[1 + \left(\frac{hv_0 Pe\, \rho_\ell^0}{p_0}\right)^{\frac{1}{2}}\right],$$

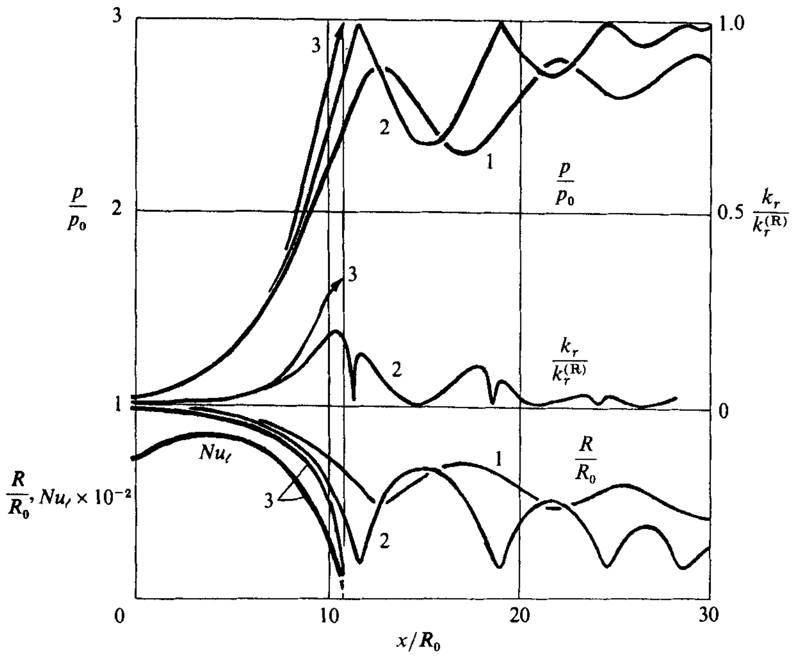

FIGURE 11. Structure of shock waves in water with vapour bubbles depending on vapour phase dispersity, $p_0 = 0.1$ MPa, $\Delta P_e = 2$, $\alpha_{v0} = 0.05$. Curves 1–3 correspond to $R_0 = 1$; 10^{-1} and 10^{-2} mm ($Pe = 61 \times 10^3$; 6.1×10^3 and 0.61×10^3) respectively. $C_{\ell*}/c_0 = 1130$ for all three cases.

whereas in the case of vapour bubble collapse in the heat and inertial regime (curve Nu_ℓ in figure 11) it decreases, tending to zero at the final stage of collapse.

Figure 11 also shows the nature of changes in the kinetic energy of small-scale radial motion of the liquid around the bubble at monotonic collapse (curve 3) and collapse accompanied by vapour bubble pulsations (curve 2). It can be seen that, when $R_0 = 10^{-1}$ mm (curve 2), the kinetic energy of small-scale radial motion of the liquid around the bubble performs damped oscillations, together with the pressure in the mixture becoming negligibly small at the final stage of the bubble collapse when it enters the thermal regime of collapse. In this case bubble contraction and expansion phases are significantly asymmetrical. During the collapse of the bubble with $R_0 = 10^{-2}$ mm (curve 3) the specific kinetic energy of small-scale motion of the liquid around the bubble k_r reaches ~ 0.3 of the limiting value $k_r^{(R)}$ when the collapse occurs in the Rayleigh regime:

$$k_r^{(R)} = \frac{\alpha_{v0} p_0 \Delta P_e}{\rho_\ell^0}.$$

During the collapse of the bubble with $R_0 = 10^{-1}$ mm (curve 2) the maximum value of $k_r/k_r^{(R)}$ does not exceed 0.2. The kinetic energy of small-scale motion during vapour bubble collapse in the thermal regime decreases and tends to zero at the final stage of collapse, and in the inertial regime it increases, as is shown in the graph, tending to a finite value. If we assume that during bubble collapse this energy is transformed into heat, the increase in the pressure due to liquid thermal expansion is insignificant. In the case of bubble collapse in water in the Rayleigh regime at $p_0 = 0.1$ MPa, $\Delta P_e = 2$, the pressure increase due to liquid heating is several thousandths of the initial pressure (Zuong Ngok Hai *et al.* 1982). If we assume that this energy of liquid radial motion around the bubble at the moment of its collapse is converted into the energy of elastic compression of the liquid (Nigmatulin *et al.* 1982), the pressure increase can become significant and, as far as the order of magnitude is concerned, is in good agreement with the experimental evidence of shock-wave enhancement (Borisov *et al.* 1977; Gel'fand *et al.* 1978).

For this final stage of collapse, the calculation becomes more complicated. Besides, at this stage the proposed model can become invalid because of the necessity of taking into account the liquid compressibility and other effects.

The behaviour of the curves in figure 11 shows that at $R \rightarrow 0$ the bubble surface velocity increases strongly, and, because of its finite thermal conductivity, the liquid fails to rapidly transport the heat liberated during vapour condensation. Accordingly, the condition $p_v = p_s(T_v) = \text{const.}$ is not satisfied. As a result, the pressure in the bubble grows, which can lead to an incomplete collapse of the bubble and its subsequent pulsations. In this case the shock wave may have a peaked oscillatory structure, as was found experimentally (Deksnis 1978; Borisov *et al.* 1982) for strong shock waves.

5. Condensation waves. Shock adiabat

Let a stationary shock wave, the equilibrium states behind which we shall denote by a superscript 1, move through a motionless, relative to the wall, equilibrium mixture towards this wall. This shock wave is incident on the wall, reflects from it and moves backwards. Let us denote by a superscript 2 the equilibrium state behind the reflected shock wave after it reaches the steady regime. The other designations remain the same. It is assumed that in front of the incident wave the medium is at rest ($v_0 = 0$), behind the incident wave it acquires velocity $v^{(1)}$, and after reflection of the wave from the wall the medium is again at rest relative to the wall ($v^{(2)} = 0$).

We shall consider waves sufficiently long that equilibrium could be established behind each wave before another wave arrives. For this to occur, it is necessary that the tube length should be much larger than the thickness of a shock wave. In this case the carrier liquid is assumed to be compressible. If the void fraction is small ($\alpha_v \lesssim \alpha_\ell$) and the conditions are far from critical, the vapour mass can be negelected compared to the carrying-liquid mass, i.e. $m_v = \alpha_v \rho_v^0 / \alpha_\ell \rho_\ell^0 \ll 1$. Thus, the presence of the vapour phase manifests itself only in the fact that the mixture is compressible as a whole. The equilibrium of bubbly vapour–liquid media is unstable. That is why when shock waves of finite intensities propagate in such media, the vapour in the bubbles condenses, and the single-phase medium – liquid – is present behind a wave. The reflected wave will propagate in the single-phase compressible liquid. It is assumed that the wave reflects from a perfectly rigid wall.

In a system containing not very small bubbles ($R \gtrsim 10^{-3}$ mm) the main dissipative mechanism is a thermal one. In this case the necessary energetic condition for conservation of the bubbly structure of a mixture of a liquid with vapour bubbles can be written in the following form:

$$c_\ell[T_s(p^{(1)}) - T_s(p_0)] \leqslant m_v l.$$

Using the Clapeyron–Clausius equation (4) the sufficient condition for the mixture bubbly structure destruction can be written in the form

$$P^{(1)} - 1 \geqslant \delta P = \frac{\epsilon_0}{a_* C_{\ell *}} m_v l \quad \left(P^{(1)} = \frac{p^{(1)}}{p^0}, \quad a_*^2 = \frac{p_0}{\rho_{\ell 0}^0}\right). \tag{25}$$

In many media this condition is satisfied for waves with very low intensities. For water at $p \sim 0.1$ MPa, $\alpha_{v0} \sim 10^{-2}$, δP is of the order of 10^{-4}. However, a decrease in wave intensity ($P^{(1)} \to +1$) leads, as is shown in figure 7, to a wave with a more smooth structure, to an increase in its thickness. That is why the equilibrium pressure behind the reflected shock wave $p^{(2)}$ due to the blurred incident shock wave will be reached only after a considerable period of time. Waves of low intensity can be considered in terms of linear theory (Trammell 1962; Nakoryakov & Shreiber 1979; Azamatov & Shagapov 1981). Below we shall consider only appreciably strong waves ($P^{(1)}-1 > \delta P$).

For the medium under consideration, the laws of conservation of masses and momentum of the mixture in the case of an incident wave with velocity $U^{(1)}$ and a reflected wave with velocity $U^{(2)}$ (relative to the wall) assume the following form (jump conditions):

$$\left.\begin{aligned} \rho_0 U^{(1)} = \rho^{(1)}(U^{(1)}-v^{(1)}); \quad p^{(1)}-p_0 = \rho_0 U^{(1)} v^{(1)}; \\ \rho^{(1)}(U^{(2)}+v^{(1)}) = \rho^{(2)}U^{(2)}; \quad p^{(2)}-p_0^{(1)} = \rho^{(1)}(U^{(2)}+v^{(1)})\,v^{(1)}; \\ \rho_0 = \alpha_{\ell 0}\rho^0_{\ell 0}(1+m_{v0}) \approx \alpha_{\ell 0}\rho^0_{\ell 0}, \quad \alpha_v^{(1)} = \alpha_v^{(2)} = 0. \end{aligned}\right\} \tag{26}$$

For waves with finite intensities ($P^{(1)}-1 > \delta P$), $\rho^{(1)}$, $\rho^{(2)}$ are the densities of the single-phase liquid behind the incident and reflected wave, respectively. At pressures $p \sim 0.1$–10 MPa the following equations of state of linear acoustics are used for them:

$$p^{(1)}-p_0 = a_\ell^2(\rho^{(1)}-\rho^0_{\ell 0}); \quad p^{(2)}-p^{(1)} = a_\ell^2(\rho^{(2)}-\rho^{(1)}). \tag{27}$$

The parameters behind the shock wave can be expressed by known parameters in front of it from (26) and (27). It is easy to express the velocity of propagation of the incident wave through its intensity and the parameters in front of it:

$$\left.\begin{aligned} U^{(1)} = U_0\left[\frac{1+\alpha_c(P^{(1)}-1)}{1+(\alpha_c/\alpha_{v0})(P^{(1)}-1)}\right]^{\frac{1}{2}}, \\ \left(\alpha_c = \frac{a_*^2}{a_\ell^2} = \frac{p_0}{\rho^0_{\ell 0}a_\ell^2}, \quad U_0^2 = \frac{p^{(1)}-p_0}{\alpha_{\ell 0}\alpha_{v0}\rho^0_{\ell 0}}\right). \end{aligned}\right\} \tag{28}$$

From (28) it follows that at $a_l \to +\infty$ (an incompressible carrier liquid) $\alpha_c \to +0$, $U^{(1)} \to U_0$. From this it is also seen that the influence of the carrier-liquid compressibility is expressed through the last cofactor, and in order that this influence be negligible, i.e. the compression of the mixture be due to the compression of the bubbles, it is necessary and sufficient that

$$\delta_v = \frac{\alpha_c}{\alpha_{v0}}(P^{(1)}-1) = \frac{p^{(1)}-p_0}{\alpha_{v0}\rho^0_{\ell 0}a_\ell^2} \ll 1. \tag{29}$$

For $\alpha_{v0} \sim 10^{-2}$, $\rho_{\ell 0} \sim 10^3$ kg/m^3, $a_l \sim 10^3$ m/s this condition is satisfied if $p^{(1)}-p_0 \lesssim 1.0$ MPa.

For the pressure behind the reflected wave we obtain the following expression

$$\left.\begin{aligned} \frac{p^{(2)}}{p^{(1)}} = 1+f(P^{(1)}); \\ f(P^{(1)}) = \frac{1+\alpha_c(P^{(1)}-1)}{\alpha_c P^{(1)}}A^{\frac{1}{2}}[(\tfrac{1}{4}A)^{\frac{1}{2}}+(1+\tfrac{1}{4}A)^{\frac{1}{2}}]; \\ A = \frac{\alpha_{v0}}{\alpha_{\ell 0}}\alpha_c(P^{(1)}-1)\,\frac{1+(\alpha_c/\alpha_{v0})(P^{(1)}-1)}{1+\alpha_c(P^{(1)}-1)}. \end{aligned}\right\} \tag{30}$$

If conditions $\delta_v \ll 1$ and $\alpha_{v0}\delta_v^{\frac{1}{2}} \ll 1$ are satisfied, the following simplified expression follows from (30):

$$\left.\begin{aligned} \frac{p^{(2)}}{p^{(1)}} &= 1+f(P^{(1)}); \\ f(P^{(1)}) &= \frac{U_f}{P^{(1)}}(P^{(1)}-1)^{\frac{1}{2}} \quad \left(U_f = \left[\frac{\alpha_{v0}}{\alpha_{\ell 0}}\frac{\rho^0_{\ell 0}a_\ell^2}{p_0}\right]^{\frac{1}{2}}\right). \end{aligned}\right\} \tag{31}$$

It shows the degree of shock-wave enhancement in a vapour–liquid mixture due to reflection from a rigid wall when an incident wave causes the complete condensation of vapour. At $\alpha_{v0} = 0$ (31) yields the known result for a low-compressible linear acoustic medium

$$\frac{p^{(2)}}{p^{(1)}} = 2-\frac{1}{P^{(1)}}. \tag{32}$$

The dependences (30) and (31) have a singularity. At

$$P^{(1)} \to +1, \frac{df}{dP^{(1)}} \sim (P^{(1)}-1)^{-\frac{1}{2}} \to +\infty,$$

i.e. when the wave intensity $(P^{(1)}-1)$ decreases to 0, the pressure in the medium way increase infinitely in the case of reflection from a rigid wall.This is explained by the fact that for a given bubbly structure of the mixture with $P^{(1)} \to +1$ the condition (25), on the basis of which (31) and (30) have been obtained, can be violated.

In the case of a liquid with bubbles of insoluble and non-condensable gas the parameters behind the reflected shock wave can also be found in terms of the dependence on the parameters in front of the wave:

$$\left.\begin{aligned} \frac{p^{(2)}}{p^{(1)}} &= 1+(P^{(1)}-1)(1-\Phi_1)[P^{(1)}(\Phi_1-\Phi_2)]^{-1}, \\ \Phi_i &= \alpha_{v0}\frac{p_0}{p^{(i)}}+\alpha_{\ell 0}\left[1+\frac{\alpha_{\ell 0}(p^{(i)}-p_0)}{\rho^0_{\ell 0}a_\ell^2}\right]^{-1}, \quad i = 1,2. \end{aligned}\right\} \tag{33}$$

Unlike the case of a liquid with vapour bubbles, in this case the bubbly structure of a mixture behind the wave is not damaged. The media behind the reflected and incident shock waves are also mixtures of a liquid with gas bubbles with the respective mean densities $\rho^{(1)} \approx \alpha_\ell^{(1)}\rho_\ell^{0(1)}, \rho^{(2)} \approx \alpha_\ell^{(2)}\rho_\ell^{0(2)}$.

We obtain the following expression for the velocity of propagation of the incident shock wave:

$$U^{(1)} = \left[\frac{p_0}{\alpha_{\ell 0}\alpha_{v0}\rho^0_{\ell 0}}P^{(1)}\right]^{\frac{1}{2}}\left[1+\frac{P^{(1)}}{P^{(1)}-1}\delta_v+O(\delta_v^2)\right]^{\frac{1}{2}}, \tag{34}$$

which was first derived by Campbell & Pitcher (1958). The problem of the velocity of shock-wave propagation in gas–liquid media was also discussed by Parkin, Gilmore & Brode (1961) who give the following expression for the shock-wave propagation velocity:

$$U^{(1)} = \left[\tfrac{1}{2}(\gamma+1)\frac{p_0}{\alpha_{v0}\rho^0_{\ell 0}}P^{(1)}\right]^{\frac{1}{2}}\left(1+\frac{\gamma-1}{\gamma+1}\frac{1}{P^{(1)}}\right)^{\frac{1}{2}}, \tag{35}$$

which at $\gamma = 1$, $\alpha_{v0} \ll 1$ coincides with (34). For comparison let us take the

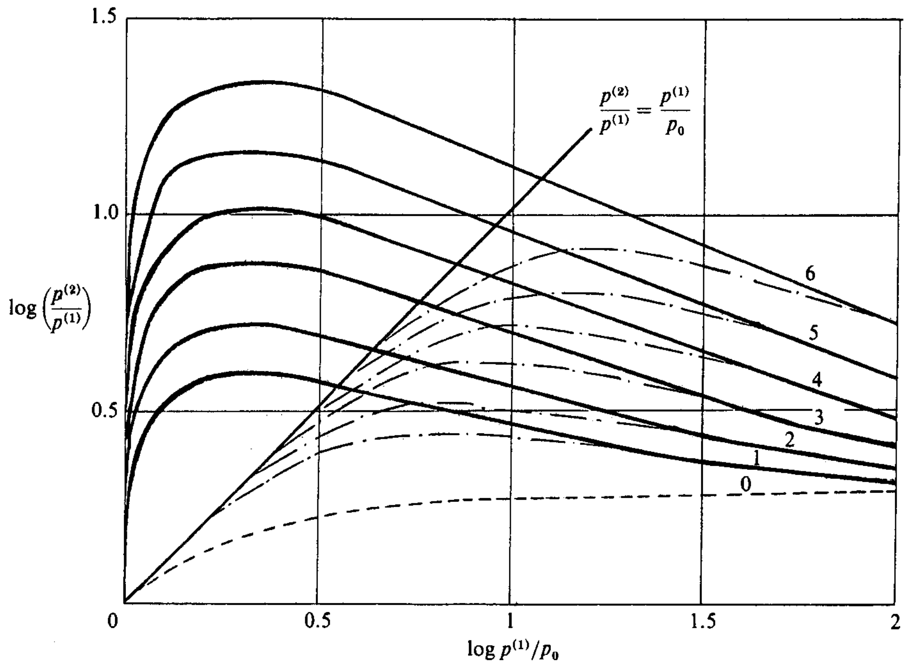

FIGURE 12. Shock adiabat of reflected waves in water with bubbles of vapour (—) and air (-·-·-). Curves 0–6 correspond to vapour (gas) volume fraction $\alpha_{v0} = 0$; 0.005; 0.01; 0.025; 0.05; 0.10 and 0.20, respectively. The dashed line corresponds to (32), with complete condensation of the vapour.

dependence of $p^{(2)}/p^{(1)}$ on $P^{(1)}$ for the mixture of an incompressible liquid and gas bubbles with constant mass, which follows from (33)

$$\frac{p^{(2)}}{p^{(1)}} = P^{(1)}. \tag{36}$$

As (31) (or (30)) shows, a considerable enhancement of the shock wave after its reflection can occur in a vapour–liquid medium. For example, for a vapour–water mixture at $p_0 = 0.1$ MPa ($\rho^0_{\ell 0} \approx 10^3$ kg/m^3, $a_\ell \approx 1500$ m/s) with a volume fraction of $\alpha_{v0} = 0.1$ we have $U_f = 50$, and after the reflection of waves with pressures $p^{(1)} = 0.2$ and 0.5 MPa, the pressures on the wall will be equal to $p^{(2)} = 5.2$ and 10.5 MPa, respectively. Besides the enhancement increases as the vapour volume fraction grows.

Figure 12 presents graphs of the enhancement of plane shock waves reflected from a rigid wall for vapour–water, air–water and linear acoustic media together with (30) or (31) (solid curves), (33) (dash-dotted curves) and (32) (dashed curve). Bubbly liquids exhibit a strong physical nonlinearity in the medium compression due to an abrupt decrease of the compressibility upon the bubble volume decrease. It is seen that a complete condensation of the vapour in the bubbles changes the structure of the mixture and its compressibility and leads to an anomalous increase in pressure in the medium during shock-wave reflection from a rigid wall.

It should be noted that in experiments with shock tubes it is not always possible to measure the pressure $p^{(2)}$ owing to the specifications of the experiment. If the length of the relaxation zone of an incident wave is large, the rarefaction wave from the high-pressure chamber of the shock tube (Nigmatulin 1982) can reach the opposite wall before the equilibrium pressure is established. In this case p_w, the maximum pressure measured on the wall, will be smaller than its expected equilibrium value $p^{(2)}$. Thus, the pressure p_w measured in experiments can depend on several factors: initial radius of the bubbles, thermophysical properties of the vapour phase, wave intensity, presence or absence of effects of bubble breakup and so on. Other conditions being equal, the value of p_w can also be influenced by the length of the high-pressure chamber of the shock tube. Note that in the case of a sufficient incident-wave intensity the bubbles break up, and the relaxation-zone length (profile blur) of the wave abruptly decreases. In this case the equilibrium maximum pressures behind the reflected waves are, as a rule, established before rarefaction waves arrive. If bubbles remain intact in a wave, the situation changes sharply, since the blurring of the incident-wave front becomes stronger.

6. Conclusion

A mathematical model to describe wave processes in liquids with vapour bubbles with allowance for the nonlinear non-steady interphase heat and mass transfer has been proposed. General regularities of the propagation of shock waves of moderate intensities and delta-pulses in these media depending on the initial conditions, mixture structure and thermophysical properties of phases have been studied for the first time on the basis of this model. It has been established that:

(i) Thermophysical properties of the phases influence the wave evolution through four dimensionless parameters $Pe, C_{\ell*}/\epsilon_0, \gamma, C_{v*}$, whereas in the case of cold liquids (no phase transition) the above influence is expressed only through two dimensionless parameters

$$R_0 D_v^{-1}\left(\frac{p_0}{\rho_\ell^0}\right)^{\frac{1}{2}}, \quad \gamma.$$

(ii) The steady structure of compression waves does exist in a bubbly vapour–liquid medium and is unique for a given wave intensity; the wave may be either monotonic or oscillatory. At $p \sim 0.1$ MPa, $R \sim 1$ mm, for most media the wave transient zone has a length of the order of dozens of centimetres.

(iii) Of all dissipative mechanisms, thermal dissipation has a radical effect on the evolution of waves in vapour–liquid bubbly media. The distance of wave transition to the steady structure in a liquid with vapour bubbles is much shorter than in a liquid with bubbles of insoluble gas, and the wave thickness is also much smaller. Unlike the case of a liquid with gas bubbles, the thickness of a steady wave in a vapour–liquid bubbly medium is determined by the thermal diffusivity of the liquid but not of the gas (vapour).

(iv) An increase in the initial static pressure of the system, wave intensity and initial bubble radius, and a decrease in the initial void fraction lead to an increase in the distance of transition to the steady structure, as well as to an enhancement of the tendency towards oscillations in a wave. In contrast to steady shock waves within which the pressure in the mixture cannot exceed the pressure behind the wave, in the case of steady waves the enhancement of the originally initiated waves is possible.

(v) In the process of the evolution of 'long' shock waves the size of the region of the perturbed motion embranced by radial pulsations of bubbles, and the number of oscillatory peaks behind the wave front increase. In this case the structure of sufficiently weak unsteady waves changes from oscillatory to monotonic, and the structure of stronger waves evolves to the limiting oscillatory configuration.

(vi) The phenomenon of the anomalous enhancement of waves reflected from a wall was explained by analysing jump conditions in a bubbly vapour–liquid medium. This phenomenon is considerably weaker in a cold gas–liquid mixture since its bubbly structure is not violated.

(vii) Comparison of the results of calculations with the experimental data has shown that the model is suitable for describing wave processes in vapour–liquid bubbly media over a wide range of wave intensities and basic parameters of the medium.

(viii) In spite of a significant deviation of the bubble shape from sphericity observed behind the wavefront, the agreement between theory and experiment is good. This testifies to a decisive influence of the volume but not shape of the bubble upon the radial motion.

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